SSTPRS 2022

Week 2: Differential Geometry

- frame field description
- vector space
- differential geometry
- curvature
- metric

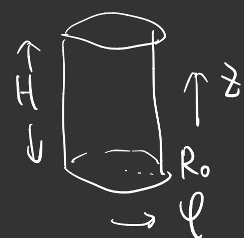
Lecture 1 : Frame field description

geometry

geo → earth

metry → measurement

<u>Physical Cylinder</u> :



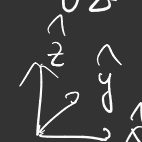
$0 \leq \varphi \leq 2\pi$

$0 \leq z \leq H$

consider $f(\varphi, z)$ its " total differential "

$$df(\varphi, z) = d\varphi \frac{\partial f}{\partial \varphi} + dz \frac{\partial f}{\partial z}$$

write the position vector in terms of



$$\vec{R}(\varphi, z) = R_0 [\cos\varphi \, \hat{x} + \sin\varphi \, \hat{y}] + z\hat{z}$$

$$\Rightarrow d\vec{R} = d\varphi \frac{\partial \vec{R}}{\partial \varphi} + dz \frac{\partial \vec{R}}{\partial z}$$

$$= d\varphi [-R_0 \sin\varphi \, \hat{x} + R_0 \cos\varphi \, \hat{y}] + dz \, \hat{z}$$

$$d\vec{R} \cdot d\vec{R} = (d\varphi)^2 [R_0^2 \sin^2\varphi + R_0^2 \cos^2\varphi] + (dz)^2$$

$$\parallel$$

$$ds^2 = (d\varphi)^2 R_0^2 + (dz)^2$$

$$\Updownarrow$$

(line element)^2

introduce "metric" : $ds^2 = (d\varphi \quad dz) \begin{pmatrix} g_{\varphi\varphi} & g_{\varphi z} \\ g_{z\varphi} & g_{zz} \end{pmatrix} \begin{pmatrix} d\varphi \\ dz \end{pmatrix}$

constant

$\Rightarrow \partial g = 0$

$\Rightarrow$ curvature & connection are vanishing

$$\Rightarrow \quad g_{\varphi\varphi} = R_0^2, \quad g_{\varphi z} = g_{z\varphi} = 0, \quad g_{zz} = 1$$

Define the inverse frame operators as

$$e_\phi = \frac{1}{R_0}\frac{\partial}{\partial\varphi}, \qquad e_z = \frac{\partial}{\partial z}$$

one can check $[e_\varphi, e_z] = 0$

Generally, the embedding of a p-dimensional surface Σ with coordinate $x^{\underline{m}}$ into the D-dim'l space is accomplished as follows.

$\vec{r}$: a vector points from some fixed origin to a point on Σ.

$$\vec{r} = \vec{X}(x) = X^{\underline{a}}(x)\, \ell^{\underline{a}}$$

D linearly independent unit vectors

gradient operator $\vec{\nabla} = \ell^{\underline{a}}\, e_{\underline{a}}{}^{\underline{m}}\, \partial_{\underline{m}}$ $\quad \underline{a} = a_1, a_2, \cdots, a_D$, $\quad \underline{m} = m_1, \cdots m_p$

differential line element $\quad d\vec{r} = dx^{\underline{m}}\, \hat{e}_{\underline{m}}{}^{\underline{a}}\, \ell^{\underline{a}}$

if we choose $\ell^{\underline{a}} \cdot \ell^{\underline{b}} = \delta^{\underline{a}\underline{b}}$ then $e_{\underline{a}}{}^{\underline{m}}$ is the inverse frame to $\hat{e}_{\underline{m}}{}^{\underline{a}}$

metric : $ds^2 = dx^{\underline{m}} g_{\underline{mn}} dx^{\underline{n}} = dx^{\underline{m}} \underbrace{e_{\underline{m}}{}^{\underline{a}}}_{\text{frame field}} dx^{\underline{n}} e_{\underline{n}}{}^{\underline{b}} \delta_{\underline{ab}}$

frame field operator $e_{\underline{a}} = e_{\underline{a}}{}^{\underline{m}} \partial_{\underline{m}}$: $\vec{\nabla} = \ell^{\underline{a}} e_{\underline{a}}$

anholonomy $c^{\underline{c}}$: $[e_{\underline{a}}, e_{\underline{b}}] = \underbrace{\epsilon_{\underline{ab}} c^{\underline{c}}}_{C_{\underline{ab}}{}^{\underline{c}}} e_{\underline{c}}$

Exercise: ① $\vec{r}(\phi, z) = R_0\left(1 - \frac{z}{H_0}\right)[\hat{x}\cos\phi + \hat{y}\sin\phi] + z\hat{z}$

"right circular cone"

$$d\vec{r} = R_0\left(1-\frac{z}{H_0}\right)d\phi\left[-\hat{x}\sin\phi + \hat{y}\cos\phi\right] - dz\left[\left(\frac{R_0}{H_0}\right)(\hat{x}\cos\phi + \hat{y}\sin\phi) - \hat{z}\right]$$

calculate $g_{\underline{m}\underline{n}}$, $e_{\underline{a}}$, $C_{\underline{a}\underline{b}}{}^{\underline{c}}$

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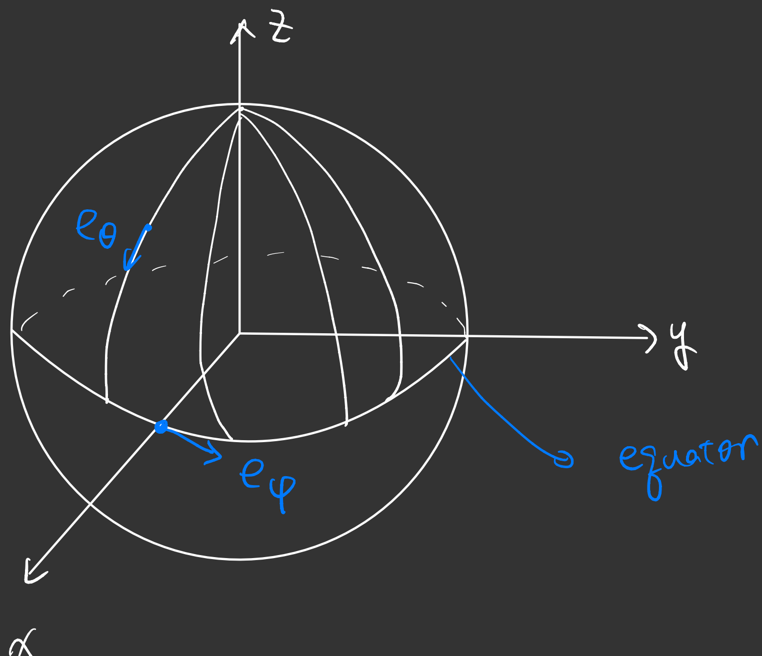
geometry interpretation:

$$f(x+\xi) = e^{\left[\xi \frac{d}{dx}\right]} f(x) \qquad \text{Taylor's Theorem}$$

$$= \left(\mathbb{1} + \xi\frac{d}{dx} + \frac{\xi^2}{2!}\frac{d^2}{dx^2} + \cdots\right) f(x)$$

$$= f(x) + \xi f'(x) + \frac{1}{2}\xi^2 f''(x) + \cdots$$



More exercise: the 2-sphere

$$\vec{r}(\theta,\varphi) = R[\hat{x}\cos\varphi\sin\theta + \hat{y}\sin\varphi\sin\theta + \hat{z}\cos\theta]$$

$$d\vec{r} = R\,d\theta[\hat{x}\cos\varphi\cos\theta + \hat{y}\sin\varphi\cos\theta - \hat{z}\sin\theta]$$
$$+ R\sin\theta\,d\varphi[-\hat{x}\sin\varphi + \hat{y}\cos\varphi]$$

$$ds^2 = d\vec{r}\cdot d\vec{r} = R^2(d\theta)^2 + R^2\sin^2\theta\,(d\varphi)^2$$

$$g_{\underline{mn}} = \begin{pmatrix} R^2 & 0 \\ 0 & R^2\sin^2\theta \end{pmatrix}$$

$$e_\theta = \frac{1}{R}\frac{\partial}{\partial\theta} = \frac{1}{R}\partial_\theta \qquad e_\phi = \frac{1}{R\sin\theta}\frac{\partial}{\partial\phi} = \frac{1}{R\sin\theta}\partial_\phi$$

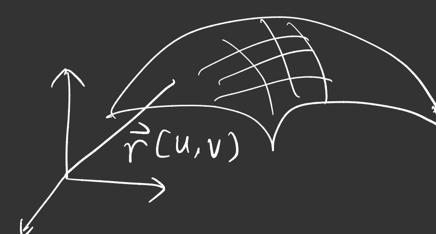
$$[e_\theta, e_\phi] = \left[\frac{1}{R}\partial_\theta, \frac{1}{R\sin\theta}\partial_\phi\right]$$

$$= \frac{1}{R^2}\left[\partial_\theta, \frac{1}{\sin\theta}\right]\partial_\phi = \frac{1}{R^2}(-1)\frac{\cos\theta}{\sin^2\theta}\partial_\phi$$

$$= -\frac{\cot\theta}{R} e_\phi$$

Let $\vec{r}(u,v)$ where u, v are generalized coordinates

and $\vec{r}(u,v)$ is a given function



$$d\vec{r} = du\frac{\partial\vec{r}}{\partial u} + dv\frac{\partial\vec{r}}{\partial v}$$

$$ds^2 = (du \quad dv)\begin{pmatrix} g_{uu} & g_{uv} \\ g_{vu} & g_{vv} \end{pmatrix}\begin{pmatrix} du \\ dv \end{pmatrix}$$

If $g_{uv} = g_{vu} = 0$ (g is diagonal) then

$$\begin{pmatrix} g_{uu} & 0 \\ 0 & g_{vv} \end{pmatrix} \xrightarrow{\text{take } \sqrt{\ }} \begin{pmatrix} \sqrt{g_{uu}} & 0 \\ 0 & \sqrt{g_{vv}} \end{pmatrix}$$

$$[e_u, e_v] = C_{uv}{}^{u} e_u + C_{uv}{}^{v} e_v \qquad \Leftarrow \begin{cases} e_u = \dfrac{1}{\sqrt{g_{uu}}} \dfrac{\partial}{\partial u} \\ e_v = \dfrac{1}{\sqrt{g_{vv}}} \dfrac{\partial}{\partial v} \end{cases}$$

Introduce the curvature

$$R \equiv \epsilon^{ab} e_a c_b - c^d c_d = e_\theta c_\phi - e_\phi c_\theta - c^\theta c_\theta - c^\phi c_\phi$$

Gaussian Mean Curvature (scalar Riemann curvature)

※ $c_\phi \equiv c_{\theta\phi}{}^{\theta}$, $c_\theta \equiv c_{\theta\phi}{}^{\phi}$

<u>Exercise:</u> calculate R for 2-sphere. $R = \frac{1}{R_0^2}$

<u>Homework:</u> 1) Torus $\vec{r}(u,v) = R_1 [(\hat{x}\cos u + \hat{y}\sin u)(1 + \rho_0 \cos v)$

$+ \rho_0 \hat{z} \sin v]$

ρ_0: constant

2) Non-orthogonal plane

$$\vec{r} = u\hat{x} + v[\hat{x}\cos\varphi_0 + \hat{y}\sin\varphi_0] \qquad \varphi_0: \text{const.}$$

calculate: $d\vec{r}$, ds^2, g, frame operators, C.

Lecture 2 : Vector Space

Review of Linear algebra:

(Basis, Components, Change of Basis, ... tensors)

Def: a R - vector space is a triplet $(V, \oplus, \odot)$

$\mathbb{R}, \mathbb{C}$

- $V = \{v_1, v_2, \dots, v_N\}$

vectors : elements of set V

- $\oplus : (V, V) \longrightarrow V$

$v_1, v_2 \longrightarrow v_1 \oplus v_2$

$- \odot : (\mathbb{R}, V) \to V$

$\lambda \in \mathbb{R}, v_1 \to \lambda \odot v_1 = \lambda v_1$

Properties: C A N I (for $+$)

commutative

$v_1 + v_2 = v_2 + v_1$

associativity

$(v_1+v_2)+v_3$

$= v_1 + (v_2+v_3)$

neutral "0"

$v + 0 = v$

inverse

$\forall v, \exists -v$

$v + (-v) = 0$

A D D U (for $\cdot$)

associativity

$\lambda_1 \cdot (\lambda_2 \cdot v)$

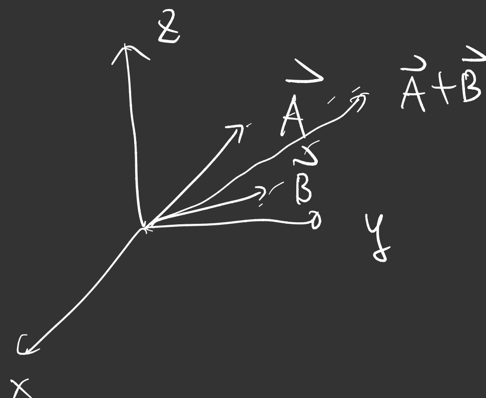
$= (\lambda_1 \cdot \lambda_2) \cdot v$

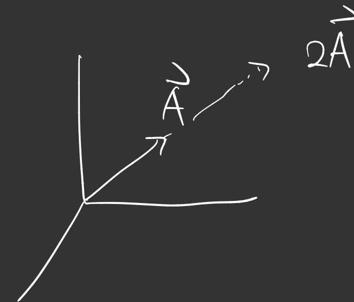
distributative

unit

$1 \cdot v = v$

Example 1) position vectors (configuration space)





2) $n \times n$ matrices

$$V = \{ A_{n\times n}, B_{n\times n}, \cdots \}$$

$\oplus$ = addition of matrices

$\odot$ = multiplication of matrices

3) $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ are also vector spaces

x $z = x + iy$, $i^2 = -1$ quaternions $h = x_1 + i x_2 + j x_3 + k x_4$

$i^2 = j^2 = k^2 = -1 \qquad ij = k$

<u>Basis</u> of $(V, +, \cdot)$ is:

$$B \subseteq V = \{v_1, \dots, v_N\}$$

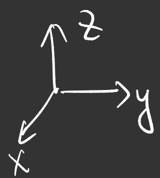
$$\| $$

$$\{e_1, e_2, \dots, e_d\}$$

1) linearly independent

$$\lambda^i e_i = 0 \Rightarrow \lambda^i = 0$$

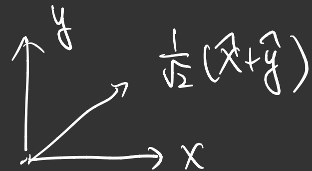
Example 1



$\hat{x}, \hat{y}, \hat{z}$ are linearly independent.

$$c_1 \hat{x} + c_2 \hat{y} + c_3 \hat{z} = 0 \Rightarrow c_1 = c_2 = c_3 = 0$$

Example 2:



$$e_1 = \hat{x}$$

$$e_2 = \frac{1}{\sqrt{2}}(\hat{x}+\hat{y})$$

$$c_1 e_1 + c_2 e_2 = 0 \Rightarrow c_1 = c_2 = 0$$

Example 3:

$$a\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + b\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + c\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + d\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = 0$$

$$\Rightarrow a = b = c = d = 0$$

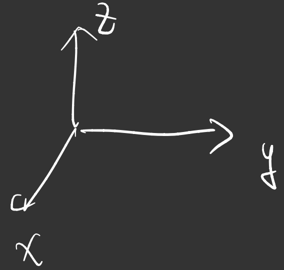
2) $\forall\ v \in V$ $\qquad \vec{v} = v^i e_i$

e_i: basis $\in B$

v^i: vector components

they are unique given the basis e_i

Example:



choose $B = \{\hat{x}, \hat{y}\}$

$e_1 = \hat{x}, \quad e_2 = \hat{y}$

Q1. are e_1, e_2 linearly independent? $\boxed{\text{Yes}}$

Q2. Can all vectors be written as linear combination of $\hat{x}, \hat{y}$? $\boxed{\text{NO}}$

Dimensionality of $V(+, \cdot)$

$$\dim(V) \equiv |B| = d \qquad B = \{e_1, e_2, \cdots, e_d\}$$

Change basis : $\{e_i\} \to \{\tilde{e}_i\}$

$$V = v^i e_i$$

$$\tilde{e}_i = \tilde{A}_i{}^j e_j$$

components of $\tilde{e}_i$ as expressed in the basis e_i

then $v = \tilde{v}^i \tilde{e}_i = v^i e_i$

$= \tilde{v}^i \tilde{A}_i{}^j e_j$

$\tilde{A}_i{}^j \tilde{A}^{-1}{}_j{}^k = (\mathbb{1})_i{}^k = \delta_i{}^k$

$\Rightarrow \tilde{v}^i \tilde{A}_i{}^j = v^j \Rightarrow \tilde{v}^k = v^j \tilde{A}^{-1}{}_j{}^k$

Dual: A vector (co-vector) in V^* can "eat" a vector $v \in V$ and $\Rightarrow$ a number

Dual Vector Space V^*

$B = \{ \epsilon^1, \epsilon^2, \cdots, \epsilon^d \}$ basis of V^*

$\omega = \epsilon^i \omega_i$ — components of ω in basis $\{\epsilon^i\}$

$\omega \in V^*$ (covector)

$$(V^*, V) \longrightarrow \mathbb{R}$$

$$\omega(v) = \epsilon^i(v)\,\omega_i = \epsilon^i(v^j e_j)\,\omega_i$$

$$= v^j \underbrace{\epsilon^i(e_j)}_{\substack{\parallel \\ c^i{}_j \in \mathbb{R}}}\,\omega_i = \omega^T \cdot C \cdot v$$

$$(\omega_1 \cdots \omega_d)\begin{pmatrix} & & \\ & C & \\ & & \end{pmatrix}\begin{pmatrix} v_1 \\ \vdots \\ v_d \end{pmatrix}$$

Exercise: $e_1 = \hat{x}$, $e_2 = \frac{1}{\sqrt{2}}(\hat{x} + \hat{y})$

$\epsilon_1 = ?$ $\quad \epsilon_2 = ?$ such that

$$\epsilon^i(e_j) = \delta^i{}_j$$

set $\epsilon_1 = (a \quad b)$ $\quad (a \quad b)\begin{pmatrix} 1 \\ 0 \end{pmatrix} = 1 \quad (a \quad b)\begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} = 0$

$\epsilon_2 = (c \quad d)$ $\quad (c \quad d)\begin{pmatrix} 1 \\ 0 \end{pmatrix} = 0 \quad (c \quad d)\begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} = 1$

$\Rightarrow \epsilon_1 = (1 \quad -1)$

$\epsilon_2 = (0 \quad \sqrt{2})$

※ dual to the unit vector

don't have to be unit vector

Change basis in V^*

$\omega = \epsilon^i \omega_i \qquad \{\epsilon^i\} \to \{\tilde{\epsilon}^i\} \qquad \tilde{\epsilon}^i = \epsilon^j B_j{}^i$

$= \tilde{\epsilon}^i \tilde{\omega}_i = \epsilon^j \underbrace{B_j{}^i \tilde{\omega}_i}_{\omega_j} \quad \Rightarrow \quad \omega_j = B_j{}^i \tilde{\omega}_i$

$\tilde{\omega}_i = B^{-1}{}_i{}^j \omega_j$

$V: \{e_i\} \to \{\tilde{e}_i\}$ $\quad v = v^i e_i \quad , \tilde{v}^i = v^j (A^{-1})_j{}^i$

$\tilde{e}_i = A_i{}^j e_j$

$V^*: \{\epsilon^i\} \to \{\tilde{\epsilon}^i\}$ $\quad \omega = \epsilon^i \omega_i , \quad \tilde{\omega}_i = (B^{-1})_i{}^j \omega_j$

$\tilde{\epsilon}^i = \epsilon^j B_j{}^i$

<u>Duality:</u> $\epsilon^i e_j = \delta^i{}_j$ we want $\tilde{\epsilon}^i \tilde{e}_j = \delta^i{}_j$

$\Rightarrow \underbrace{\epsilon^k}\, B_k{}^i A_j{}^l \underbrace{e_l} = B_k{}^i A_j{}^l \delta^k{}_l = A_j{}^k B_k{}^i = \delta^i{}_j$

$\Rightarrow B = A^{-1}$

Tensors

v^i , ω_i

(p,q) tensor is a generalization of vector or covector

with p indices of the vector type : i

q indices of the covector type : j

$$T^{i_1 i_2 \cdots i_p}{}_{j_1 j_2 \cdots j_q}$$

e.g. $(p=1,\ q=0)$ T^i components of a vector

$(p=0,\ q=1)$ T_j components of a covector

$(p=1,\ q=1)$ $T_i{}^j$ matrix

$(p=0, q=2)$ T_{ij} e.g. metric g_{ij}

$$T_{\hat{j}_1 \hat{j}_2 \cdots \hat{j}_q}{}^{i_1 i_2 \cdots i_p} \longrightarrow \tilde{T}_{\hat{j}_1 \cdots \hat{j}_q}{}^{i_1 \cdots i_p}$$

$$= A_{\hat{j}_1}{}^{k_1} A_{\hat{j}_2}{}^{k_2} \cdots A_{\hat{j}_q}{}^{k_q} \tilde{T}_{k_1 \cdots k_q}{}^{l_1 \cdots l_p}$$

$$(A^{-1})_{l_1}{}^{\hat{i}_1} \cdots (A^{-1})_{l_p}{}^{\hat{i}_p}$$

in previous note, $A = B^{-1}$

recall $\tilde{\omega}_i = (B^{-1})_i{}^{\hat{j}} \omega_j$

Lecture 3: Differential Geometry

physics (SR, GR) space(time) : stage where physics takes place

core assumption : spacetime = continous "manifold"

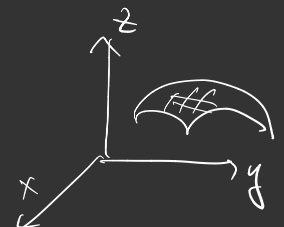
SR v.s. GR

static dynamic (Einstein Eq.)

$g_{\mu\nu}$

Ways to describe curved manifolds

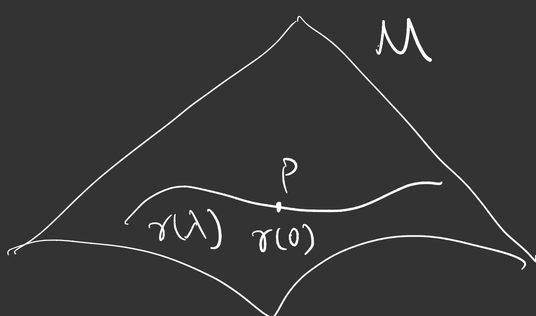
1) extrinsic approach : embedding space



$$\vec{r} = f_1(u,v)\,\hat{x} + f_2(u,v)\,\hat{y} + f_3(u,v)\,\hat{z}$$

2) instrinsic approach

- no embedding space (no outsides)
- no tangent space (no arrows)



$\gamma(0) = p$

$\gamma(\lambda) =$ trajectory of the surface M.

$$\begin{tikzcd} \mathbb{R} \arrow[r, "\gamma"] \arrow[rr, bend right, "f(\gamma(\lambda))"'] & M \arrow[r, "f"] & \mathbb{R} \end{tikzcd}$$

$$X_{\gamma,p} f \equiv \left[f(\gamma(\lambda)) \right]' \Big|_{\lambda=0}$$ derivative w.r.t. λ

$$= \frac{\partial f}{\partial x^m}\Big|_p \underbrace{\frac{d\gamma^m(\lambda)}{d\lambda}\Big|_{\lambda=0}}_{V^m} = V^m \frac{\partial f}{\partial x^m}\Big|_p \quad \forall f$$

$$\Rightarrow \quad X_{\gamma,p} \equiv V^m \frac{\partial}{\partial x^m}$$ "the differential derivative operator at p along curve γ"

in Diff Geo. $X_{\gamma,p}$ is called "tangent vector" of curve γ at p.

- no arrows, no directions
- a differential operator
- tangent "plane" $\rightarrow$ tangent space $\left\{ \frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2}, \cdots \right\}$

is a vector space

$\frac{\partial}{\partial x^m}$ form a basis

- change of basis = change of coordinates

$$X = v^m \frac{\partial}{\partial x^m} = \tilde{v}^m \frac{\partial}{\partial \tilde{x}^m} = \tilde{v}^m \frac{\partial x^n}{\partial \tilde{x}^m} \frac{\partial}{\partial x^n}$$

$$\Rightarrow v^n = \tilde{v}^m \frac{\partial x^n}{\partial \tilde{x}^m} \quad \Rightarrow \tilde{v}^m = v^n \left(\frac{\partial \tilde{x}^m}{\partial x^n} \right)$$

- Dual vector space $\omega = dx^m \, \omega_m$

$\quad\quad$ 1-form

$$\frac{\partial x^k}{\partial x^m} = \delta^k_m$$

$$\tilde{\omega}_m = \left(\frac{\partial x^n}{\partial \tilde{x}^m} \right) \omega_n$$

- (p, q)-Tensor in Diff Geo.

$$\tilde{T}_{j_1 \cdots j_q}{}^{i_1 \cdots i_p} = \left(\frac{\partial x^{k_1}}{\partial \tilde{x}^{j_1}} \right) \cdots \left(\frac{\partial x^{k_q}}{\partial \tilde{x}^{j_q}} \right) T_{k_1 \cdots k_q}{}^{l_1 \cdots l_p} \left(\frac{\partial \tilde{x}^{i_1}}{\partial x^{l_1}} \right) \cdots \left(\frac{\partial \tilde{x}^{i_p}}{\partial x^{l_p}} \right)$$

Lecture 4: Curvature

* parallel transport, connection, and covariant derivative

* curvature

Motivate parallel transport

the question that we want to understand:

- how tensors are transported along a curve.

- two vectors defined at different points cannot be compared naively with each other.

Flat space $\mathbb{R}^m$ vector $V = V^\mu e_\mu$ $V^\mu(x)$

x^μ $\mu = 1, \cdots, m$

we define the derivative

$$\frac{\partial V^\mu}{\partial x^\nu} = \lim_{\Delta x \to 0} \frac{V^\mu(x^\nu + \Delta x) - V^\mu(x^\nu)}{\Delta x^\nu}$$

subtract: transport $V^\mu(x)$ to $x + \Delta x$ without change

$\downarrow$ parallel transport

Call $\tilde{V}\big|_{x+\Delta x}$ denote a vector $V|_x$ parallel transported to $x + \Delta x$

We demand the components

① $\tilde{V}^\mu(x+\Delta x) - V^\mu(x) \quad \propto \Delta x$

② $\widetilde{(V^\mu + W^\mu)}(x+\Delta x) = \tilde{V}^\mu(x+\Delta x) + \tilde{W}^\mu(x+\Delta x)$

we take $\tilde{V}^\mu(x+\Delta x) = V^\mu(x) - \Delta x^\nu (\quad)_\nu{}^\mu$

① ✓ ②: LHS $= V^\mu + W^\mu - \Delta x^\nu (\,V+W\,)_\nu{}^\mu$

RHS $= V^\mu - \Delta x^\nu (\,V\,)_\nu{}^\mu + W^\mu - \Delta x^\nu (\,W\,)_\nu{}^\mu$

$$\Rightarrow \quad \tilde{V}^\mu(x+\Delta x) = V^\mu(x) - V^\lambda(x)\,\Gamma^\mu{}_{\nu\lambda}(x)\,\Delta x^\nu$$ "connection"

<u>covariant derivative</u> $= \lim_{\Delta x^\nu \to 0} \dfrac{V^\mu(x+\Delta x) - \tilde{V}^\mu(x+\Delta x)}{\Delta x^\nu}$

covariant derivative of V w.r.t. x^ν

$$= \lim_{\Delta x^\nu \to 0} \frac{V^\mu(x+\Delta x) - \tilde{V}^\mu(x+\Delta x)}{\Delta x^\nu} e_\mu$$

$$\left(\frac{\partial V}{\partial x^\nu} = \frac{\partial V^\mu e_\mu}{\partial x^\nu} \right.$$

$$e_\mu = \frac{\partial}{\partial x^\mu}$$

$$= \lim_{\Delta x^\nu \to 0} \frac{V^\mu(x+\Delta x) - V^\mu(x) + V^\lambda(x) \Gamma^\mu{}_{\nu\lambda}(x) \Delta x^\nu}{\Delta x^\nu} e_\mu$$

$$= \left(\frac{\partial V^\mu}{\partial x^\nu} + V^\lambda \Gamma^\mu{}_{\nu\lambda} \right) e_\mu$$

is a vector at $x + \Delta x$

<u>covariant derivative</u>

$$\nabla_\nu V^\mu = \frac{\partial V^\mu}{\partial x^\nu} + V^\lambda \Gamma^\mu{}_{\nu\lambda}$$

$$\nabla_\nu \omega_\kappa = ?$$

Idea:
- $\omega_\kappa V^\kappa = f(x)$ – scalar
- $\nabla_\mu f(x) = \frac{\partial f}{\partial x^\mu}$

$$\Rightarrow \nabla_\mu (\omega_\kappa V^\kappa) = (\nabla_\mu \omega_\kappa) V^\kappa + \omega_\kappa (\nabla_\mu V^\kappa)$$

$$= \underbrace{(\nabla_\mu \omega_\kappa)}_{} V^\kappa + \omega_\kappa \left(\cancel{\frac{\partial V^\kappa}{\partial x^\mu}} + V^\lambda \Gamma^\kappa{}_{\mu\lambda} \right)$$

$$= \frac{\partial \omega_\kappa}{\partial x^\mu} V^\kappa + \cancel{\omega_\kappa \frac{\partial V^\kappa}{\partial x^\mu}}$$

$$\Rightarrow (\nabla_\mu \omega_\kappa) V^\kappa = \left(\frac{\partial \omega_\kappa}{\partial x^\mu} - \omega_\lambda \Gamma^\lambda{}_{\mu\kappa} \right) V^\kappa \qquad \forall V^\kappa$$

$$\Rightarrow \quad \nabla_\mu \omega_\kappa = \frac{\partial \omega_\kappa}{\partial x^\mu} - \omega_\lambda \Gamma^\lambda{}_{\mu\kappa}$$

generalize to (p,q) - tensor:

$$\nabla_\mu \left(T_{j_1 \cdots j_q}{}^{i_1 \cdots i_p} \right) = \frac{\partial}{\partial x^\mu} \left(T_{j_1 \cdots j_q}{}^{i_1 \cdots i_p} \right) + T_{j_1 \cdots j_p}{}^{\lambda i_2 \cdots i_p} \Gamma^{i_1}{}_{\mu\lambda}$$

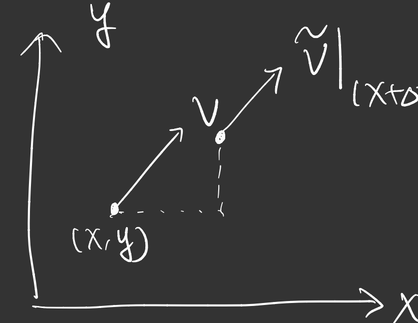
$$+ \cdots + T_{j_1 \cdots j_q}{}^{i_1 \cdots i_{p-1} \lambda} \Gamma^{i_p}{}_{\mu\lambda}$$

$$- T_{\lambda \cdots j_p}{}^{i_1 \cdots i_p} \Gamma^{\lambda}{}_{\mu j_1} - \cdots - T_{j_1 \cdots \lambda}{}^{i_1 \cdots i_p} \Gamma^{\lambda}{}_{\mu j_q}$$

Example: $\mathbb{R}^2$

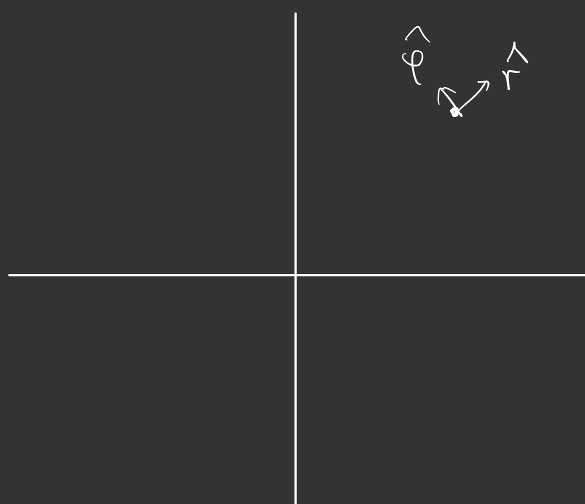
Cartesian coordinate (x, y)

$$\Gamma^{\mu}{}_{\nu\lambda} = 0$$



$$\tilde{v}\big|_{(x+\Delta x,\, y+\Delta y)} = v\big|_{(x,y)}$$

polar coordinate (r, φ)



$$\vec{r} = r\cos\varphi\,\hat{x} + r\sin\varphi\,\hat{y}$$

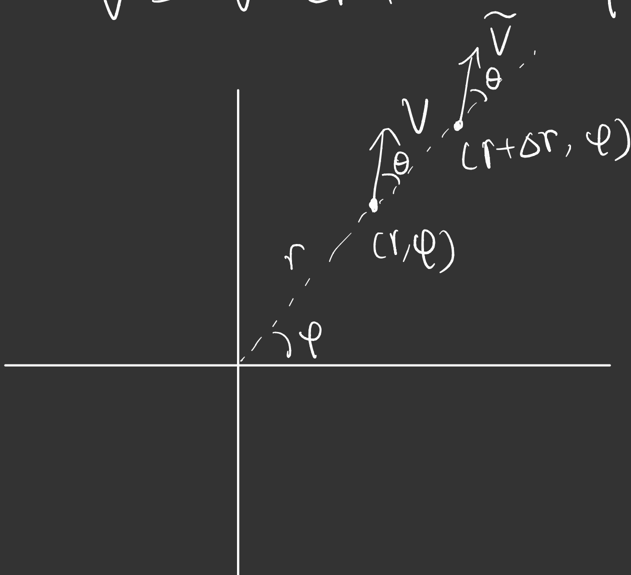
$$d\vec{r} = (\cos\varphi\,\hat{x} + \sin\varphi\,\hat{y})\,dr + r(-\sin\varphi\,\hat{x} + \cos\varphi\,\hat{y})\,d\varphi$$

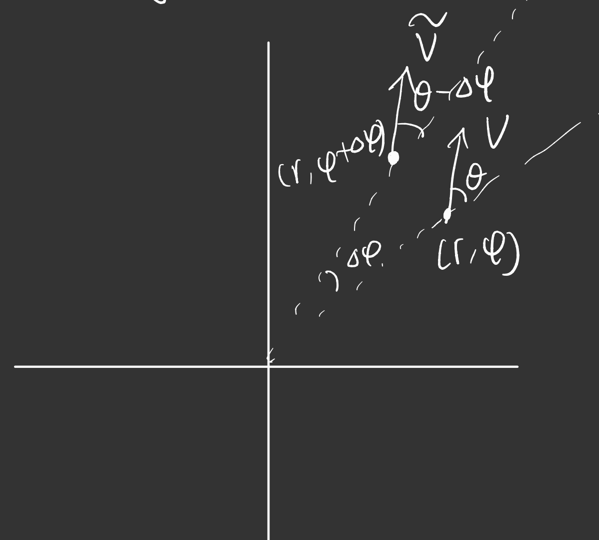
$$d\vec{r}\cdot d\vec{r} = (\cos\varphi\,dr - r\sin\varphi\,d\varphi)^2 + (\sin\varphi\,dr + r\cos\varphi\,d\varphi)^2$$

$$= dr^2 + r^2\,d\varphi^2$$

$$V = V^r e_r + V^\varphi e_\varphi$$

vector defined at (r, φ)





$$\left.\begin{aligned} V^r &= V\cos\theta \\ V^\phi &= \frac{V\sin\theta}{r} \end{aligned}\right\} \Leftrightarrow V = \sqrt{g(V \cdot V)}$$

$$= \sqrt{V^r V^r g_{rr} + V^\phi V^\phi g_{\phi\phi}}$$

$$= \sqrt{V^r V^r + V^\phi V^\phi r^2}$$

$$\tilde{V}^r = V^r \Rightarrow \Gamma^r{}_{\nu\lambda} = 0$$

$$\tilde{V}^\phi = \frac{V\sin\theta}{r+\Delta r} = \frac{r}{r+\Delta r} V^\phi$$

$$= \frac{1}{1+\frac{\Delta r}{r}} V^\phi$$

$$\sim \left(1 - \underbrace{\frac{\Delta r}{r}}\right) V^\phi$$

$$\Rightarrow \Gamma^\phi{}_{r\phi} = \frac{1}{r}$$

$$\tilde{V}^r = V\cos(\theta - \Delta\varphi)$$

$$= V\cos\theta\cos\Delta\varphi + V\sin\theta\sin\Delta\varphi$$

$$\sim V\cos\theta + V\sin\theta\,\Delta\varphi$$

$$= V^r + r V^\phi \Delta\varphi$$

$$\tilde{V}^\phi = \frac{V\sin(\theta - \Delta\varphi)}{r}$$

$$= \frac{V\sin\theta\cos\Delta\varphi - V\cos\theta\sin\Delta\varphi}{r}$$

$$\sim \frac{V\sin\theta}{r} - \frac{V\cos\theta}{r}\Delta\varphi$$

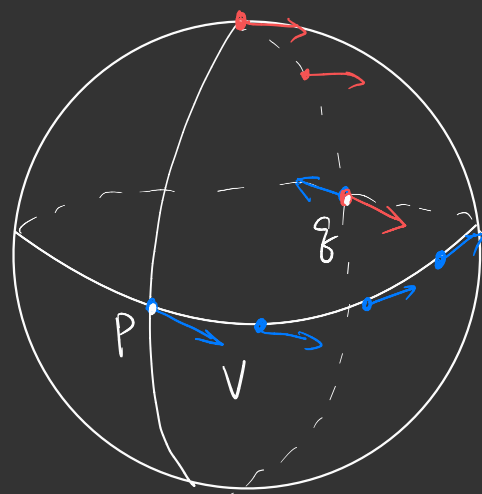
$$= V^\phi - \frac{V^r}{r}\Delta\varphi$$

$$\Rightarrow \Gamma^r{}_{\phi\phi} = -r \ ; \quad \Gamma^\phi{}_{\phi r} = \frac{1}{r}$$

- Here we assume that the norm of a vector is invariant under parallel transport
- $\Gamma^{\mu}{}_{\lambda\nu} = \Gamma^{\mu}{}_{\nu\lambda}$

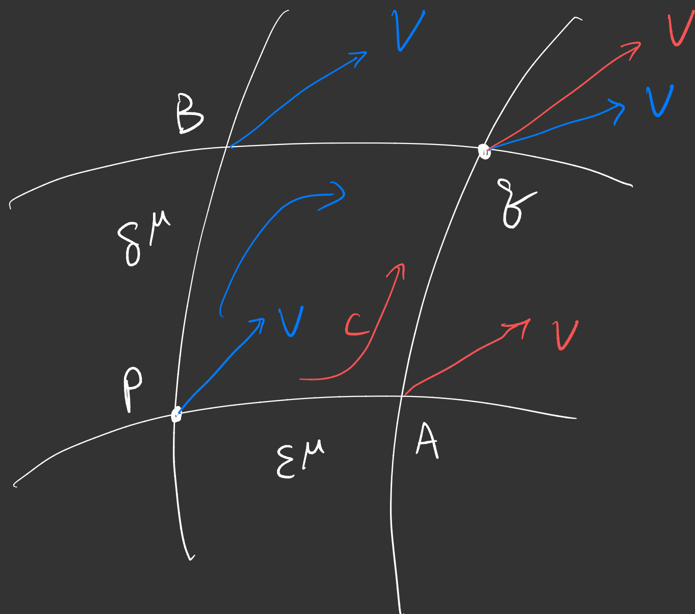
Levi-Civita connection

Curvature



2-sphere

V parallel transported along a great circle if the angle V makes with the great circle is kept fixed



curvature measures the difference when we parallel transport a vector at p to q along different curves.

- intrinsic $\Rightarrow$ does not depend on the special coordinates chosen.

$$\tilde{V}_c^{\mu}(A) = V_0^{\mu} - V_0^{\lambda}\, \Gamma^{\mu}{}_{\nu\lambda}(p)\, \varepsilon^{\nu}$$

$$\tilde{V}_c^{\mu}(q) = \tilde{V}_c^{\mu}(A) - \tilde{V}_c^{\lambda}(A)\, \Gamma^{\mu}{}_{\nu\lambda}(A)\, \delta^{\nu}$$

$$\Rightarrow \tilde{V}^{\mu}_{c}(\gamma) = V_0^{\mu} - V_0^{\lambda}\Gamma^{\mu}{}_{\nu\lambda}(p)\,\varepsilon^{\nu} - \left[V_0^{\lambda} - V_0^{\rho}\Gamma^{\lambda}{}_{\nu\rho}(p)\,\varepsilon^{\nu}\right]$$

$$\times \underbrace{\Gamma^{\mu}{}_{\sigma\lambda}(A)}_{\parallel}\,\delta^{\sigma}$$

$$\Gamma^{\mu}{}_{\sigma\lambda}(p) + \partial_{\tau}\Gamma^{\mu}{}_{\sigma\lambda}(p)\,\varepsilon^{\tau} \qquad \partial_{\tau} = \frac{\partial}{\partial x^{\tau}}$$

$$= V_0^{\mu} - V_0^{\lambda}\Gamma^{\mu}{}_{\nu\lambda}(p)\,\varepsilon^{\nu} - V_0^{\lambda}\Gamma^{\mu}{}_{\sigma\lambda}(p)\,\delta^{\sigma}$$

$$- V_0^{\lambda}\,\partial_{\nu}\Gamma^{\mu}{}_{\sigma\lambda}(p)\,\varepsilon^{\nu}\delta^{\sigma} + V_0^{\lambda}\Gamma^{\rho}{}_{\nu\lambda}(p)\,\Gamma^{\mu}{}_{\sigma\rho}(p)\,\varepsilon^{\nu}\delta^{\sigma}$$

$$\tilde{V}^{\mu}_{c'}(\gamma) = V_0^{\mu} - V_0^{\lambda}\Gamma^{\mu}{}_{\nu\lambda}(p)\,\delta^{\nu} - V_0^{\lambda}\Gamma^{\mu}{}_{\nu\lambda}(p)\,\varepsilon^{\nu}$$

$$- V_0^{\lambda}\left[\partial_{\sigma}\Gamma^{\mu}{}_{\nu\lambda}(p) - \Gamma^{\rho}{}_{\sigma\lambda}(p)\,\Gamma^{\mu}{}_{\nu\rho}\right]\varepsilon^{\nu}\delta^{\sigma}$$

Difference : $\tilde{V}_{c'}^{\mu} - \tilde{V}_{c}^{\mu} = V_0^{\lambda} \Big[\partial_\nu \Gamma^{\mu}{}_{\sigma\lambda}(p) - \Gamma^{\rho}{}_{\nu\lambda}(p)\Gamma^{\mu}{}_{\sigma\rho}(p)$

$$- \partial_\sigma \Gamma^{\mu}{}_{\nu\lambda} + \Gamma^{\rho}{}_{\sigma\lambda}(p)\Gamma^{\mu}{}_{\nu\rho}(p)\Big] \varepsilon^{\nu}\delta^{\sigma}$$

$$\equiv V_0^{\lambda} \underline{R^{\mu}{}_{\lambda\nu\sigma}} \varepsilon^{\nu}\delta^{\sigma}$$

Riemann Curvature Tensor $R^{\mu}{}_{\lambda\nu\sigma}$, which is a (1,3) tensor

$$R^{\mu}{}_{\lambda\nu\sigma} \equiv \Gamma^{\mu}{}_{\nu\rho}(X)\Gamma^{\rho}{}_{\sigma\lambda}(X) - \Gamma^{\mu}{}_{\sigma\rho}(X)\Gamma^{\rho}{}_{\nu\lambda}(X)$$
$$- \partial_\sigma \Gamma^{\mu}{}_{\nu\lambda}(X) + \partial_\nu \Gamma^{\mu}{}_{\sigma\lambda}(X)$$

<u>Properties</u>:

- $R^{\mu}{}_{\lambda\nu\sigma} = -R^{\mu}{}_{\lambda\sigma\nu}$ "antisymmetric in ν and σ"

- works in any D dimensions

$\mu, \nu, \sigma, \lambda = 0, 1, \cdots, D-1$

- In 2D, $\mu, \lambda, \nu, \sigma = 0, 1$

$R^{\mu}{}_{\lambda\nu\sigma} \to \epsilon_{\nu\sigma} R^{\mu}{}_{\rho} \to \epsilon_{\nu\sigma} \epsilon^{\mu}{}_{\rho} R$

reduce to $(0,0)$ tensor R (Gauss mean curvature)

- Ricci tensor: $R_{\rho\sigma} \equiv R^{\mu}{}_{\rho\mu\sigma}$

- curvature is independent on the choice of Γ connections.

Lecture 5: Metric

principle curvature

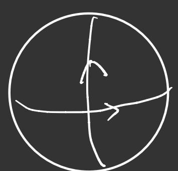
k_1 is the curvature along γ_1

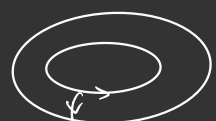
k_2 – – – – γ_2

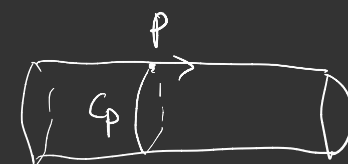
2D $\quad R^{\mu}{}_{\nu\rho\sigma}(X) \longrightarrow R(X) = k_1(X)\, k_2(X)$

Gauss mean curvature

$$R(X)\big|_p = k_1\big|_p \cdot k_2\big|_p$$







$$R_{cylinder} = k_1 \cdot k_2 = 0$$

curvature of straight line @ p

curvature of the circle C_p

Metric $g_{\mu\nu}$: symmetric $(0,2)$ - tensor $\quad g_{\mu\nu} = g_{\nu\mu}$.

<u>Riemannian manifolds</u> $\quad (X^\mu, g_{\mu\nu})$

Let's assume that someone gives you a metric $g_{\mu\nu}$.

$$\underset{(1,0)}{V^\mu} \longrightarrow V^\mu g_{\mu\nu} = \underset{(0,1)}{(V)_\nu}$$

lowering the index via the metric $g_{\mu\nu}$

$$\nabla_\mu V^\rho = \frac{\partial V^\rho}{\partial x^\mu} + V^\lambda \Gamma^\rho{}_{\mu\lambda}$$

$$\nabla_\mu (\underbrace{V^\rho g_{\rho\sigma}}_{(0,1)}) = \cancel{\frac{\partial}{\partial x^\mu}(V^\rho g_{\rho\sigma})} - \cancel{(V^\rho g_{\rho\lambda})\Gamma^\lambda{}_{\mu\sigma}}$$

$$\text{LHS} = (\nabla_\mu V^\rho)\, g_{\rho\sigma} + V^\rho (\nabla_\mu g_{\rho\sigma})$$

$$= \cancel{\frac{\partial V^\rho}{\partial x^\mu} g_{\rho\sigma}} + V^\lambda \Gamma^\rho{}_{\mu\lambda}\, g_{\rho\sigma} + \cancel{V^\rho \frac{\partial g_{\rho\sigma}}{\partial x^\mu}} - V^\rho g_{\lambda\sigma} \Gamma^\lambda{}_{\mu\rho}$$

$$- \cancel{V^\rho g_{\rho\lambda} \Gamma^\lambda{}_{\mu\sigma}}$$

$$\Rightarrow \quad V^\lambda \Gamma^\rho{}_{\mu\lambda}\, g_{\rho\sigma} - V^\rho g_{\lambda\sigma} \Gamma^\lambda{}_{\mu\rho} = 0$$

$$V^\lambda \left[\Gamma_{\sigma\mu\lambda} - \Gamma_{\sigma\mu\lambda} \right] = 0 \qquad \checkmark$$

there is an opportunity that $\nabla_\mu g_{\rho\sigma} = 0$ (Metricity Condition)

$$\Rightarrow \quad \boxed{\frac{\partial g_{\rho\sigma}}{\partial x^\mu} = g_{\lambda\sigma} \Gamma^\lambda{}_{\mu\rho} + g_{\rho\lambda} \Gamma^\lambda{}_{\mu\sigma}} \qquad (*)$$

$\Rightarrow$ Levi-civita Connection $\Gamma^{\lambda}{}_{\mu\rho}$.

From $(*)$: ① solve for Γ

② $R^{\mu}{}_{\rho\nu\sigma}(\Gamma, \partial\Gamma) = R^{\mu}{}_{\rho\nu\sigma}(g, \partial g)$

<u>Inverse metric:</u> $g^{\mu\nu}$ symmetric $(2,0)$

raising index: $\underset{(0,1)}{g^{\mu\nu}\omega_{\nu}} = \underset{(1,0)}{\omega^{\mu}}$

$R^{\mu}{}_{\nu\rho\sigma}(x)\, g_{\mu\lambda}(x) = R_{\lambda\nu\rho\sigma}(x)$ $(0,4)$ tensor

- $R_{\mu\nu\rho\sigma} = R_{\rho\sigma\mu\nu}$
- $R_{\mu\nu\rho\sigma} = -R_{\nu\mu\rho\sigma}$
- $R_{\mu\nu\rho\sigma} = -R_{\mu\nu\sigma\rho}$

example: Euclidean spacetime $g_{\mu\nu} = g^{\mu\nu} = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$

Minkowski : $g_{\mu\nu} = g^{\mu\nu} = \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$