Celestial Dual Superconformal Symmetry, MHV Amplitudes and Differential Equations

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Abstract

Celestial and momentum space amplitudes for massless particles are related to each other by a change of basis provided by the Mellin transform. Therefore properties of celestial amplitudes have counterparts in momentum space amplitudes and vice versa. In this work, we study the celestial avatar of dual superconformal symmetry of $\mathcal{N}=4$ Yang-Mills theory. We also analyze various differential equations known to be satisfied by celestial n-point tree-level MHV amplitudes and identify their momentum space origins.

Motivations

The quest for flat space holography has recently received a boost owing to the realization that scattering amplitudes in 4D flat spacetime can be recast as correlation functions of a 2D conformal field theory living on the celestial sphere [1]–[3]. Then the celestial CFT (CCFT) becomes a potential candidate for a holographic description of the flat space S-matrix. A path towards a better understanding of CCFTs involves translating well understood aspects of momentum space amplitudes into statements about celestial correlators, as well as mapping momentum space amplitudes onto the celestial sphere.

In this work, we look at this problem from the both sides: we study the celestial avatar of the dual superconformal symmetry of $\mathcal{N}=4$ Yang-Mills; we also identify the momentum space origins of various differential equations satisfied by celestial n-point tree level MHV amplitudes.

Celestial Dual Superconformal Symmetry

Let us first rewrite the expression for the generators $K^{\alpha\dot{\alpha}}$ and $\mathcal{S}^A_\alpha$ given in [4] in a more compact form

$$\mathcal{K}^{\alpha\dot{\alpha}} = -\sum_{i<j}\left(\tilde{\lambda}^{\dot{\alpha}}_i\lambda^\alpha_j D_{j,i} + \lambda^\alpha_i\tilde{\lambda}^{\dot{\alpha}}_i\right),$$

$$\mathcal{S}^A_\alpha = -\sum_{i<j}\left(\lambda_{j,\alpha}\eta^A_i D_{j,i} + \lambda_{i,\alpha}\eta^A_i\right).$$

where we have made use of momentum conservation and also introduced the operator

$$D_{i,j} \;=\; \lambda^\alpha_j\frac{\partial}{\partial\lambda^\alpha_i} \;-\; \tilde{\lambda}_{i,\dot{\beta}}\frac{\partial}{\partial\tilde{\lambda}_{j,\dot{\beta}}} \;-\; \sum_A \eta^A_i\frac{\partial}{\partial\eta^A_j},$$

Let $\mathcal{O}$ be an operator acting on the amplitude. The corresponding operator $\tilde{\mathcal{O}}$, which acts on the celestial amplitude is defined by

$$\tilde{\mathcal{O}}\tilde{\mathcal{A}}_n := \int\left(\prod_{i=1}^n\frac{d\omega_i}{\omega_i}\omega_i^{\Delta_i}\right)\mathcal{O}\mathcal{A}_n.$$

Then the operator $\tilde{\mathcal{K}}^{\alpha\dot{\alpha}}$ and $\tilde{\mathcal{S}}^A_\alpha$ act on the celestial amplitude as

$$\begin{aligned}\tilde{\mathcal{K}}^{\alpha\dot{\alpha}} =& \sum_{i<j}\left\{\begin{pmatrix}1 & \bar{z}_i\\ z_j & \bar{z}_iz_j\end{pmatrix}\left[2\epsilon_ie^{\frac{\partial}{\partial\Delta_i}}\left(\Delta_j+J_j-z_{ij}\frac{\partial}{\partial z_j}\right)+2\epsilon_je^{\frac{\partial}{2\partial\Delta_i}+\frac{\partial}{2\partial\Delta_j}}\sum_A\eta^A_j\frac{\partial}{\partial\eta^A_i}\right.\right.\\ &\left.\left.-2\epsilon_je^{\frac{\partial}{\partial\Delta_j}}\left(\Delta_i-J_i+\bar{z}_{ij}\frac{\partial}{\partial\bar{z}_i}\right)\right]-2\epsilon_ie^{\frac{\partial}{\partial\Delta_i}}\begin{pmatrix}1 & \bar{z}_i\\ z_i & z_i\bar{z}_i\end{pmatrix}\right\}\\ \tilde{\mathcal{S}}^A_\alpha =& \sqrt{2}\sum_{i<j}\left\{\begin{pmatrix}-z_j\\ 1\end{pmatrix}\left[\epsilon_i\eta^A_ie^{\frac{\partial}{2\partial\Delta_i}}\left(\Delta_j+J_j-z_{ij}\frac{\partial}{\partial z_j}\right)+\epsilon_je^{\frac{\partial}{2\partial\Delta_j}}\eta^A_i\sum_B\eta^B_j\frac{\partial}{\partial\eta^B_i}\right.\right.\\ &\left.\left.-\epsilon_j\eta^A_ie^{\frac{\partial}{\partial\Delta_j}-\frac{\partial}{2\partial\Delta_i}}\left(\Delta_i-J_i+\bar{z}_{ij}\frac{\partial}{\partial\bar{z}_i}\right)\right]-\epsilon_i\eta^A_ie^{\frac{\partial}{2\partial\Delta_i}}\begin{pmatrix}-z_i\\ 1\end{pmatrix}\right\}.\end{aligned}$$

Differential Equations

- **The celestial tree-level MHV n-point amplitude** is given by the Mellin transform of the amplitude w.r.t. to ω_i [3], [5]

$$\begin{aligned}\tilde{\mathcal{M}}_n(J_i,\Delta_i,z_i,\bar{z}_i) \;=&\; \int\left[\prod_{i=1}^n\frac{d\omega_i}{\omega_i}\omega_i^{\Delta_i}\right]\mathcal{M}_n(h_i,\omega_i,z_i,\bar{z}_i)\\ :=&\; \mathcal{N}(z_i,\bar{z}_i)\,\delta\left(\sum_i(\Delta_i-1)\right)F\left(x_{a,b},\Delta_i\right),\end{aligned}$$

  where $F\left(x_{a,b},\Delta_i\right)$ is the Aomoto-Gelfand hypergeometric function. We find that momentum conservation and $GL(n-4)$ transformations reduce to the well-known first-order defining PDEs of AG function.

- **Momentum conservation**: the total momentum celestial operator is $\tilde{\mathbb{P}}^\mu=\sum_{i=1}^n\epsilon_i\,q_i^\mu e^{\frac{\partial}{\partial\Delta_i}}$ and we define 4 vectors v_b^μ s.t. $v_b^\mu\,\epsilon_a\,q_{a\mu}=-Ux_{a,b}$, $b=\{n-3,\,n-2,\,n-1,\,n\}$.

$$\tilde{\mathbb{P}}^\mu\tilde{\mathcal{M}}_n=0\;\Rightarrow\;\sum_{i=1}^n\epsilon_i\,v_{b\mu}\,q_i^\mu e^{\frac{\partial}{\partial\Delta_i}}\tilde{\mathcal{M}}_n=0\;\Rightarrow\;\sum_{a=1}^{n-4}x_{a,b}\frac{\partial F}{\partial x_{a,b}}=\alpha_bF$$

- **$GL(n-4)$ transformations:** using the momentum conserving delta function we can solve for arbitrary 4 ω's and these are equivalent representations up to $GL(n-4)$ transformations. This property of $\tilde{\mathcal{M}}_n$ gives rises to

$$\alpha_aF+\sum_{b=n-3}^n x_{a,b}\frac{\partial F}{\partial x_{a,b}} \;=\; -F\;\;,\;\;1\le a\le n-4$$

- **Generalized Banerjee-Ghosh (BG) equation**
  We derive momentum space generalizations of the differential equations found in [6] by connecting them to the behaviour of amplitudes under BCFW shifts:

$$\lambda_i\;\to\;\hat{\lambda}_i\;=\;\lambda_i+z\,\lambda_j\qquad\tilde{\lambda}_j\;\to\;\hat{\tilde{\lambda}}_j\;=\;\tilde{\lambda}_j-z\,\tilde{\lambda}_i.$$

  For infinitesimal z, this shift is implemented on $\mathcal{M}_n$ by the $D_{i,j}$ operator we introduced before.

$$D_{i,j}\,\mathcal{M}_n \;=\; \mathcal{M}_n\left(-\frac{\langle i-1,j\rangle}{\langle i-1,i\rangle}-\frac{\langle i+1,j\rangle}{\langle i+1,i\rangle}+4\frac{\langle j,t\rangle}{\langle i,t\rangle}\delta_{i,s}+4\frac{\langle j,s\rangle}{\langle i,s\rangle}\delta_{i,t}\right)$$

  Mapping this to the celestial sphere and taking $J_i=+$, we get

$$\begin{aligned}&\left[-\left(\Delta_i+z_{ij}\frac{\partial}{\partial z_i}\right)\;+\;\frac{z_{i-1,j}}{z_{i-1,i}}\;+\;\frac{z_{i+1,j}}{z_{i+1,i}}\;-\;1\right]\tilde{\mathcal{M}}_n\\ &\quad+\;\epsilon_i\epsilon_j\left(\Delta_j-J_j-1+\bar{z}_{ji}\frac{\partial}{\partial\bar{z}_j}\right)e^{\frac{\partial}{\partial\Delta_i}-\frac{\partial}{\partial\Delta_j}}\tilde{\mathcal{M}}_n\;=\;0\end{aligned}$$

  which generalizes the color-stripped BG equation.

- **Connect to BG equation:** without loss of generality, we choose $i=1$. After some manipulation, the color-stripped BG equation can be brought to the form

$$\begin{aligned}&\left[\left(\alpha_1+1+\sum_{b=n-3}^n x_{1,b}\frac{\partial}{\partial x_{1,b}}\right)-\sum_b\frac{\epsilon_2}{\epsilon_1}\left(x_{2,b}+\bar{z}_{1,2}\frac{\partial x_{2,b}}{\partial\bar{z}_2}\right)\left(\frac{\partial}{\partial x_{1,b}}-\frac{\partial}{\partial x_{2,b}}e^{\frac{\partial}{\partial\Delta_1}-\frac{\partial}{\partial\Delta_2}}\right)\right.\\ &\left.\qquad-\frac{\epsilon_2}{\epsilon_1}e^{\frac{\partial}{\partial\Delta_1}-\frac{\partial}{\partial\Delta_2}}\left(\alpha_2+1+\sum_b x_{2,b}\frac{\partial}{\partial x_{2,b}}\right)\right]F=0\end{aligned}$$

  which shows that it reduces to combinations of the hypergeometric equations. The orange term is identically zero based on the integral representation of F.

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