Celestial Amplitudes and Light Transforms

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Based on 2203.04255 with Luke Lippstreu, Marcus Spradlin, Akshay Yelleshpur Srikant, and Anastasia Volovich
and 2206.08875 with Shounak De, Akshay Yelleshpur Srikant, and Anastasia Volovich

1. Overview

The quest for flat space holography has recently received a boost owing to the realization that scattering amplitudes in 4D flat spacetime can be recast as correlation functions of a 2D conformal field theory living on the celestial sphere [1, 2]. Then the celestial CFT (CCFT) becomes a potential candidate for a holographic description of the flat space S-matrix. In generic CFTs, the OPE coefficients are related to three-point functions and four-point functions contain information about the spectrum of the theory, which can be deduced by means of the conformal block decomposition. In CCFT, three- and (tree-level) four-point correlators make these relationships opaque due to the distributional nature of their correlators. It was shown in [3] that certain three-point correlators involving light-ray operators take the form of standard three-point CFT correlators.

In this poster, we

- present the four-point correlator of two gluon light-ray operators and two gluon primaries from the four-gluon celestial amplitude in $(2,2)$ signature spacetime. The correlator is non-distributional and allows us to verify that light-ray operators appear in the OPE of two gluon primaries. We also carry out a conformal block decomposition of the terms involving the exchange of gluon operators.
- present the correlator of four gluon light-ray operators in celestial CFT. We find that it is described by Fox H-functions and generalized I-functions of multiple variables.

2. Preliminary

Celestial amplitudes in (2,2) signature Any (non-zero) null four-vector p^μ in $(2,2)$ signature can be uniquely parameterized as

$$p^\mu = \epsilon\,\omega\,(1+z\bar z, z+\bar z, z-\bar z, 1-z\bar z)\ , \tag{1}$$

where $\epsilon=\pm1, \omega>0$, and z and $\bar z$ are independent real variables.

- In $(1,3)$ signature: ϵ would indicate whether p^μ describes an incoming or outgoing particle.
- In $(2,2)$ signature: ϵ labels different Poincaré patches.

Note that it transforms covariantly under $SL(2,\mathbb{R})\times SL(2,\mathbb{R})$ conformal transformations:

$$\epsilon\to\epsilon\,\mathrm{sgn}((cz+d)(\bar c\bar z+\bar d))\ ,\quad z\to(az+b)/(cz+d)\ ,\quad \bar z\to(\bar a\bar z+\bar b)/(\bar c\bar z+\bar d) \tag{2}$$

Celestial and momentum space amplitudes for massless particles are related to each other by a change of basis provided by the Mellin transform:

$$\left\langle \mathcal{O}^{\epsilon_1}_{\Delta_1,J_1}(z_1,\bar z_1)\cdots\mathcal{O}^{\epsilon_n}_{\Delta_n,J_n}(z_n,\bar z_n)\right\rangle = \left(\prod_{i=1}^n\int_0^\infty d\omega_i\,\omega_i^{\Delta_i-1}\right)\mathcal{A}_n(\epsilon_i,\omega_i,z_i,\bar z_i) \tag{3}$$

Correlator of 4 gluon operators The tree-level, color-ordered, four-gluon amplitude is given by the Parke-Taylor formula and its corresponding celestial amplitude obtained by Mellin transform is

$$\begin{aligned}&\left\langle \mathcal{O}^{\epsilon_1}_{\Delta_1,-}(z_1,\bar z_1)\mathcal{O}^{\epsilon_2}_{\Delta_2,-}(z_2,\bar z_2)\,\mathcal{O}^{\epsilon_3}_{\Delta_3,+}(z_3,\bar z_3)\mathcal{O}^{\epsilon_4}_{\Delta_4,+}(z_4,\bar z_4)\right\rangle = \pi\,\delta(\beta)\,\delta(z-\bar z)\,X\left(z_{ij},\bar z_{ij}\right)\\ &\times\ \mathrm{sgn}\left(\frac{z}{z-1}\right)|z|^3\,|1-z|^{1-\Delta_2-\Delta_3}\,\Theta\left(-\epsilon_1\epsilon_4\frac{z_{24}\bar z_{24}}{z_{12}\bar z_{12}}z\right)\Theta\left(\epsilon_2\epsilon_4\frac{z_{34}\bar z_{34}}{z_{23}\bar z_{23}}\frac{1-z}{z}\right)\Theta\left(\epsilon_3\epsilon_4\frac{z_{24}\bar z_{24}}{z_{23}\bar z_{23}}(z-1)\right)\end{aligned} \tag{4}$$

where $X(z_{ij},\bar z_{ij}) = \frac{1}{|z_{12}|^{h_1+h_2}|z_{34}|^{h_3+h_4}}\left|\frac{z_{24}}{z_{14}}\right|^{h_{12}}\left|\frac{z_{14}}{z_{13}}\right|^{h_{34}}\frac{1}{|\bar z_{12}|^{\bar h_1+\bar h_2}|\bar z_{34}|^{\bar h_3+\bar h_4}}\left|\frac{\bar z_{24}}{\bar z_{14}}\right|^{\bar h_{12}}\left|\frac{\bar z_{14}}{\bar z_{13}}\right|^{\bar h_{34}}$, $h_i=(\Delta_i+J_i)/2$, $\bar h_i=(\Delta_i-J_i)/2$, $h_{ij}=h_i-h_j$, $\bar h_{ij}=\bar h_i-\bar h_j$, the cross ratio $z=(z_{12}z_{34})/(z_{13}z_{24})$, and $\Theta(x)$ denotes the Heaviside step function.

Light transforms The definitions of the light transforms are summarized as below.

Light Transform	Definition	Conformal Weight
Anti-holomorphic	$\bar{\mathbf{L}}[\mathcal{O}_{h,\bar h}](z,\bar z)=\int_{\mathbb{R}}\frac{d\bar z'}{\lvert\bar z'-\bar z\rvert^{2-2\bar h}}\,\mathcal{O}_{h,\bar h}(z,\bar z')$	$(h,1-\bar h)$
Holomorphic	$\mathbf{L}[\mathcal{O}_{h,\bar h}](z,\bar z)=\int_{\mathbb{R}}\frac{dz'}{\lvert z'-z\rvert^{2-2h}}\,\mathcal{O}_{h,\bar h}(z',\bar z)$	$(1-h,\bar h)$

Step functions & spacetime signatures The step functions appearing in (4) make light transform integrals complicated due to the independence of z and $\bar z$ in $(2,2)$ spacetime.

$(1,3)\to$ Euclidean CFT	$(2,2)\to$ Lorentzian CFT
$\bar z$ is the complex conjugate of z	z and $\bar z\in\mathbb{R}$ are independent
	sgn$(z_{ij}\bar z_{ij})\ >\ 0$: spacelike
sgn$(z_{ij}\bar z_{ij})\ \geq\ 0$	sgn$(z_{ij}\bar z_{ij})\ =\ 0$: null-separated
	sgn$(z_{ij}\bar z_{ij})\ <\ 0$: timelike

Our resolution is based on the fact that

$$\sum_{\epsilon_i=\pm}\Theta\left(-\epsilon_1\epsilon_4\frac{z_{24}\bar z_{24}}{z_{12}\bar z_{12}}z\right)\Theta\left(\epsilon_2\epsilon_4\frac{z_{34}\bar z_{34}}{z_{23}\bar z_{23}}\frac{1-z}{z}\right)\Theta\left(\epsilon_3\epsilon_4\frac{z_{24}\bar z_{24}}{z_{23}\bar z_{23}}(z-1)\right) = 2\ , \tag{5}$$

we define the following object and we will study the light transforms of it.

$$\left\langle\mathcal{O}_{\Delta_1,-}(z_1,\bar z_1)\mathcal{O}_{\Delta_2,-}(z_2,\bar z_2)\,\mathcal{O}_{\Delta_3,+}(z_3,\bar z_3)\mathcal{O}_{\Delta_4,+}(z_4,\bar z_4)\right\rangle := \sum_{\epsilon_i=\pm}\left\langle\mathcal{O}^{\epsilon_1}_{\Delta_1,-}(z_1,\bar z_1)\mathcal{O}^{\epsilon_2}_{\Delta_2,-}(z_2,\bar z_2)\,\mathcal{O}^{\epsilon_3}_{\Delta_3,+}(z_3,\bar z_3)\mathcal{O}^{\epsilon_4}_{\Delta_4,+}(z_4,\bar z_4)\right\rangle \tag{6}$$

3. Correlator of 2 light-ray gluons and 2 gluons

The correlator of two light-ray gluon operators and two gluon operators can be described by the Gauss hypergeometric functions. For example, consider $z,\bar z\in[0,1]$,

$$\begin{aligned}&\left\langle\bar{\mathbf{L}}[\mathcal{O}_{\Delta_1,-}](z_1,\bar z_1)\bar{\mathbf{L}}[\mathcal{O}_{\Delta_2,-}](z_2,\bar z_2)\,\mathcal{O}_{\Delta_3,+}(z_3,\bar z_3)\mathcal{O}_{\Delta_4,+}(z_4,\bar z_4)\right\rangle\\ &= -\ \pi\,\delta(\beta)\,X\left(z_{ij},\bar z_{ij}\right)\left(\frac{z}{1-z}\right)|z\bar z|^{1-\frac{\Delta_1+\Delta_2}{2}}\\ &\qquad\left\{|z-1|^{1-\Delta_3}\,{}_2F_1\left[1-\Delta_2,\Delta_3-1,\Delta_3+\Delta_4-2,\frac{z-\bar z}{z-1}\right]C(\Delta_3-1,\Delta_4-1)\right.\\ &\qquad\left.+\,|z-1|^{\Delta_4-2}|\bar z-z|^{\Delta_1+\Delta_2-1}\,{}_2F_1\left[2-\Delta_4,\Delta_1,\Delta_1+\Delta_2,\frac{z-\bar z}{z-1}\right]C(\Delta_1-1,\Delta_2-1)\right\}\end{aligned} \tag{7}$$

where $C(a,b)=B(a,b)+B(a,1-a-b)+B(b,1-a-b)\ ,\ B(a,b)=\frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$.
Note that the $|\bar z-z|^{\Delta_1+\Delta_2-1}$ term contributes a bulk point singularity.

Leading OPE structures via collinear limit Note that in (7) there is no $\delta(z-\bar z)$ and we can read off the leading OPE structures via evaluating the collinear limits. For example, we can read off the two-gluon-operator-OPE as below, which matches with [4].

$$\mathcal{O}_{\Delta_3,+}(z_3,\bar z_3)\mathcal{O}_{\Delta_4,+}(z_4,\bar z_4)\sim\frac{C(\Delta_3-1,\Delta_4-1)}{z_{34}}\mathcal{O}_{\Delta_3+\Delta_4-1,+}(z_4,\bar z_4)+\frac{\bar{\mathbf{L}}[\mathcal{O}_{\Delta_3+\Delta_4-1,+}](z_4,\bar z_4)}{z_{34}|\bar z_{34}|^{\Delta_3+\Delta_4-3}} \tag{8}$$

Conformal Block Decompositions For the violet term in (7), we compute its conformal block decomposition as

$$\text{violet term} = \sum_{n=0}^\infty\sum_{k=n}^\infty a_{k,n}\,k^{21}_{34}\left[\frac{\Delta_3+\Delta_4}{2}+k,\frac{\Delta_3+\Delta_4}{2}+n-1\right]\ , \tag{9}$$

where the coefficients are

$$\begin{aligned}a_{k,n}&=C(\Delta_3-1,\Delta_4-1)\,\frac{(1-\Delta_1)_n(1-\Delta_2)_n(\Delta_3-1)_n(\Delta_4-1)_n}{n!\,(\Delta_3+\Delta_4+n-3)_n(\Delta_3+\Delta_4-2)_{2n}}\\ &\times\sum_{m=0}^{k-n}\frac{(1-\Delta_2+n)_m(\Delta_4-1+n)_m}{(\Delta_3+\Delta_4+2n-2)_{2m}}\frac{(2-\Delta_1+n+m)_{k-n-m}(\Delta_3+n+m)_{k-n-m}}{(\Delta_3+\Delta_4-1+2n+2m)_{2k-2n-2m}}\end{aligned} \tag{10}$$

and $k^{21}_{34}[h,\bar h]$ are the usual conformal blocks [5]

$$k^{21}_{34}[h,\bar h] = {}_2F_1[h-h_{12},h+h_{34},2h,z]\,\bar z^{\bar h-\bar h_3-\bar h_4}{}_2F_1[\bar h-\bar h_{12},\bar h+\bar h_{34},2\bar h,\bar z]\,. \tag{11}$$

4. Correlator of 4 light-ray gluon operators

We consider the correlator involving two holomorphic and two anti-holomorphic light-ray operators:

$$\begin{aligned}&\left\langle\bar{\mathbf{L}}[\mathcal{O}_{\Delta_1,-}](z_1,\bar z_1)\,\bar{\mathbf{L}}[\mathcal{O}_{\Delta_2,-}](z_2,\bar z_2)\,\mathbf{L}[\mathcal{O}_{\Delta_3,+}](z_3,\bar z_3)\,\mathbf{L}[\mathcal{O}_{\Delta_4,+}](z_4,\bar z_4)\right\rangle\\ &=\pi\,\delta(\beta)\,X\left(z_{ij},\bar z_{ij}\right)\left|\frac{z}{\bar z}\right|^{\frac{\Delta_1+\Delta_2-2}{2}}|1-z|^{2-\Delta_2-\Delta_4}\,\mathcal{G}(z,\bar z)\end{aligned} \tag{12}$$

where $\mathcal{G}(z,\bar z)$ function can be organized as

$$\mathcal{G}(z,\bar z) = \int_{-\infty}^\infty\frac{dx}{x(x-1)}\mathcal{F}_1(z,x)\,\mathcal{F}_2(x,\bar z)\sim\int_{-\infty}^\infty dx\left(\prod_i|x-a_i|^{\alpha_i}\right){}_2F_1(\cdots)\,{}_2F_1(\cdots) \tag{13}$$

Here, both $\mathcal{F}_1(z,x)$ and $\mathcal{F}_2(x,\bar z)$ are four-marked-point integrals

$$\mathcal{F}_1(z,x)=\int_{-\infty}^\infty dy\,|y-z|^{\Delta_4-1}|x-y|^{\Delta_3-1}|y-1|^{\Delta_2-2}\,, \tag{14}$$

$$\mathcal{F}_2(x,\bar z)=\int_{-\infty}^\infty dt\,|x-t|^{\Delta_1-1}|1-t|^{\Delta_4-2}|\bar z-t|^{\Delta_2-1}\,. \tag{15}$$

and basically Gauss Hypergeometric functions. Our strategy to evaluate the $\mathcal{G}(z,\bar z)$ function:

1. Use Mellin-Barnes representation of the Gauss hypergeometric function to extract the x-integral

$${}_2F_1[a,b,c;-|z|] = \frac{1}{2\pi i}\frac{\Gamma(c)}{\Gamma(a)\Gamma(b)}\int_{-i\infty}^{i\infty}ds\,\frac{\Gamma(-s)\Gamma(a+s)\Gamma(b+s)}{\Gamma(c+s)}|z|^s \tag{16}$$

2. Do the x-integral first
3. Realize the remaining multivariable Mellin-Barnes-type integrals as the Fox H-function $\mathsf{H}(x,y)$ or Generalized I-functions $\mathsf{I}(z_1,\ldots,z_r)$.

$$\mathsf{H}(x,y) = \frac{1}{(2\pi i)^2}\int_{-i\infty}^{+i\infty}ds\int_{-i\infty}^{+i\infty}dt\,\phi_1(s+t)\,\phi_2(s)\,\phi_3(t)\,x^{-s}\,y^{-t} \tag{17}$$

$$\mathsf{I}(z_1,\ldots,z_r) = \frac{1}{(2\pi i)^r}\int_{-i\infty}^{+i\infty}\ldots\int_{-i\infty}^{+i\infty}ds_1\ldots ds_r\,\phi(s_1,\ldots,s_r)\,\theta_1(s_1)\ldots\theta_r(s_r)\,z_1^{s_1}\ldots z_r^{s_r} \tag{18}$$

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