SSTPRS 2022.

Week 1: group theory

- 3D rotation, generators, commutators
- Lie group, Lie algebra.
- SU(3)
- symmetry & invariants.

Lecture 1: 3D Rotations

consider " generator " denoted by L_3 :

define its actions on the coordinate (x, y, z) as

$$\begin{cases} L_3 x = -iy \\ L_3 y = ix \\ L_3 z = 0 \end{cases}$$

now we wish to prove that this operator generates a rotation about z- direction : let γ be an angle, consider the object

$$R_3(\gamma) \equiv \exp[-i\gamma L_3]$$

$$\Rightarrow \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = R_3(\gamma) \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} R_3(\gamma)\, x \\ R_3(\gamma)\, y \\ R_3(\gamma)\, z \end{pmatrix}$$

Evaluate each of these by using the definitions:

$$\left(\text{Recall: } e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \right)$$

$$R_3(\gamma)\, x = \exp[-i\gamma L_3]\, x$$

$$\overset{\text{series expansion}}{=} \left[1 - i\gamma L_3 + \frac{1}{2}(-i\gamma L_3)^2 + \cdots \right] x$$

$$= \left[x - \gamma y - \frac{1}{2}\gamma^2 x + \frac{1}{3!}\gamma^3 y + \cdots \right]$$

$$= x\cos\gamma - y\sin\gamma.$$

group assignments

Similarly, we have
$$\begin{cases} R_3(\gamma)\, y = x\sin\gamma + y\cos\gamma \\ R_3(\gamma)\, z = z \end{cases}$$

$$\Rightarrow R_3(\gamma)\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x\cos\gamma - y\sin\gamma \\ x\sin\gamma + y\cos\gamma \\ z \end{pmatrix}$$

$$= \begin{pmatrix} \cos\gamma & -\sin\gamma & 0 \\ \sin\gamma & \cos\gamma & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$\circledast$

$R_3(\gamma)$ is represented by a 3X3 matrix

<u>Idea:</u>

Number

$(1, 2, 3)$

$R_3(\gamma)$

<u>Representation</u>

Numeral

$$\begin{pmatrix} 1, & 2, & 3, & \cdots \\ \mathrm{I}, & \mathrm{II}, & \mathrm{III}, & \cdots \\ 一, & 二, & 三, & \cdots \end{pmatrix}$$

$$\begin{pmatrix} \cos\gamma & -\sin\gamma & 0 \\ \sin\gamma & \cos\gamma & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Group assignments:

consider

$$R_1(\alpha) \equiv \exp[-i\alpha L_1] \qquad \begin{cases} L_1 x = 0 \\ L_1 y = iz \\ L_1 z = -iy \end{cases}$$

$$R_2(\beta) \equiv \exp[-i\beta L_2] \qquad \begin{cases} L_2 x = -iz \\ L_2 y = 0 \\ L_2 z = ix \end{cases}$$

evaluate their actions.

Homework: calculate

(1)

$$L_1 L_2 \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -y \\ 0 \\ 0 \end{pmatrix}$$

(2)

$$L_2 L_1 \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ -x \\ 0 \end{pmatrix}$$

Lecture 2: commutator, Lie algebra, Lie group.

$$L_1 L_2 \begin{pmatrix} x \\ y \\ z \end{pmatrix} - L_2 L_1 \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -y \\ x \\ 0 \end{pmatrix}$$

$$(L_1 L_2 - L_2 L_1) \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -y \\ x \\ 0 \end{pmatrix} \qquad L_3 \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} iy \\ -ix \\ 0 \end{pmatrix}$$

define $\boxed{[\hat{A}, \hat{B}] \equiv \hat{A}\hat{B} - \hat{B}\hat{A} \quad \text{"commutator"}}$

operators

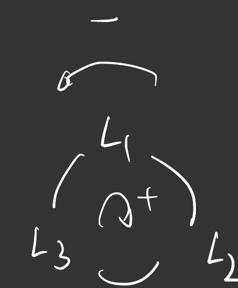
$$\Rightarrow [L_1, L_2] = iL_3$$

(vector $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ is arbitrary, suggests this equation)

$[L_2, L_3] = ?$ guess : iL_1 ✓

$[L_1, L_3] = ?$ guess : $-iL_2$

→ why? → cyclic



"set" : $SO(3) = \{L_1, L_2, L_3\}$

any two elements of ↗ , take $[L_i, L_j] = i\epsilon_{ijk}L_k$

$\Rightarrow$ $SO(3)$ is closed on $[\ ,\ ]$

calculation 1: $[[L_1, L_2], L_3] = 0 \sim [L_3, L_3]$

2: $[[L_2, L_3], L_1] = 0$

3: $[[L_3, L_1], L_2] = 0$

$$\boxed{[[L_1, L_2], L_3] + [[L_2, L_3], L_1] + [[L_3, L_1], L_2] = 0}$$

"Jacobi identity"

Continous Symmetry Group.

$\{T\} = \{T_1, \ldots, T_n\}$

Lie Algebra: step 1: a set of generators $\{T\}$

step 2: let $T_i \in \{T\}$, $T_j \in \{T\}$

then $[T_i, T_j] \in \{T\}$.

step 3: Jacobi Identity. $\forall\, T_i, T_j, T_k \in \{T\}$

$$[[T_i, T_j], T_k] + \text{cyclic} = 0$$

※ trivial closure : $T_i = T_j$

Symmetry: Lie Group $\longleftarrow$ exponentiate Lie Algebra.

$$\exp[i(\theta_1 T_1 + \theta_2 T_2 + \cdots + \theta_n T_n)]$$

$$= R_I(\theta_1, \theta_2, \cdots, \theta_n) \sim \text{general operation}$$

2x2 matrices:

(reps of operations in 2D)

$$\sigma_1 = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\sigma_2 = \frac{1}{2}\begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}$$

$$\sigma_3 = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$[\sigma_i, \sigma_j] = i\epsilon_{ijk}\sigma_k$$

(rotation of spinors)

Recall: $[L_i, L_j] = i\epsilon_{ijk}L_k$
(rotate x, y, z)

Question: do these form a Lie algebra?

check closure, Jacobi identity.

$\{\sigma_1, \sigma_2, \sigma_3\}$ and $\{L_1, L_2, L_3\}$ are different representations of the same mathematic idea.

Last time, we calculate explicit forms of R_1, R_2, R_3.

rotations along x, y, z directions.

Now, start from R_i, get generators:

e.g. $$R_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha \\ 0 & \sin\alpha & \cos\alpha \end{pmatrix}$$

expand: $$R_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1-\frac{\alpha^2}{2} & -\alpha+\frac{\alpha^3}{6} \\ 0 & \alpha-\frac{\alpha^3}{6} & 1-\frac{\alpha^2}{2} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \alpha \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$+ \alpha^2 \begin{pmatrix} 0 & 0 & 0 \\ 0 & -\frac{1}{2} & 0 \\ 0 & 0 & -\frac{1}{2} \end{pmatrix} + \alpha^3 \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \frac{1}{6} \\ 0 & -\frac{1}{6} & 0 \end{pmatrix}$$

$$+ O(\alpha^4)$$

infintesimal α:

$$R_1 = \mathbb{I}_3 + \alpha \underbrace{\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}}_{\text{call } \ell_1} + O(\alpha^2)$$

$\Downarrow$

recover $-iL_1$

recall: $L_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}$

※ $R_1 \equiv \exp[-i\alpha L_1] = \mathbb{I} - i\alpha L_1$

$\ell_1 = -iL_1$ $\{\ell_i\}$ is another representation.

$[\ell_1, \ell_2] = \ell_3$

$[\ell_2, \ell_3] = \ell_1$

$[\ell_3, \ell_1] = \ell_2$

$\ell_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}$ $\ell_3 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

$\ell_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}$

$$[l_1, l_2] = (-i)^2 [L_1, L_2] = (-1) i L_3 = -i L_3 = l_3$$

<u>Groups</u> $(G, \cdot)$

G → set elements.

$\cdot$ → bilinear operation

to define a group, we need:

1) a set of elements $G = \{g, \tilde{g}, \cdots\}$

2) " $\cdot$ " : group multiplication which will take $g, \tilde{g} \in G$

$$(g \cdot \tilde{g}) \in G.$$

Example: $N = \{1, 2, 3, \cdots\}$

$$\text{"}\cdot\text{"} = +$$

<u>properties:</u>

I): closure $\quad g, \tilde{g} \in G, \quad g \cdot \tilde{g} \in G.$

II): associativity $\quad (g \cdot \tilde{g}) \cdot \hat{g} = g \cdot (\tilde{g} \cdot \hat{g})$

III): existence of identity: $e \in G.$

so that $\quad e \cdot g = g \cdot e = g, \ \forall g \in G.$

IV): existence of inverse: $\forall g \in G$

$\exists$ a unique element $g^{-1} \in G.$

s.t. $\quad g \cdot g^{-1} = g^{-1} \cdot g = e$

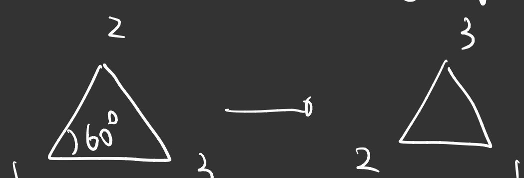
※ $\mathbb{N}$ is not a group. $\mathbb{Z}$ is a group under addition.

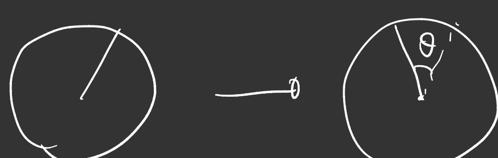
Two types of group:

- (finite) discrete group $G=\{g_1, g_2, \cdots, g_N\}$
- (continous) Lie group. $G=\{g(\alpha)\}$

Continous variable.

Example: symmetric group.





$R_3(\gamma)$:

matrix multiplication

Representation : $R_3(\gamma_2) \circ R_3(\gamma_1) = R_3(\gamma_1 + \gamma_2)$

$$R_3(\gamma) \xrightarrow{\text{vector rep}} R_3(\gamma) = \begin{pmatrix} \cos\gamma & -\sin\gamma & 0 \\ \sin\gamma & \cos\gamma & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

3x3 matrices that act on 3D position vector.

$$\xrightarrow{\text{spinor rep}} R_3(\gamma) = \exp\left[-i\gamma \tfrac{1}{2}\sigma_3\right]$$

2x2 matrices act on the space of spinors $\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$

For $G = \{g(\alpha)\}$

consider infinitesimal α: $g(\alpha) = g(0) + \alpha^I T_I + \alpha^I \alpha^J T_{IJ} + \cdots$

$\alpha = \{\alpha^1, \alpha^2, \cdots, \alpha^N\} = \alpha^I \qquad I = 1, \cdots, N$

group parameters

$$\left(\alpha^I T_I = \sum_{I=1}^{N} \alpha^I T_I \qquad \text{Einstein summation convention} \right)$$

* we choose α^I s.t. $g(0) = \mathbb{1}$

* $T_{IJ} = T_{JI}$ why? $\alpha^I \alpha^J = \alpha^J \alpha^I$

$$\alpha^I \alpha^J T_{IJ} = \alpha^J \alpha^I T_{JI} = \alpha^I \alpha^J T_{JI}$$

$$\alpha^I \alpha^J (T_{IJ} - T_{JI}) = 0 \quad \forall \alpha \Rightarrow T_{IJ} = T_{JI}$$

Note that we have for a rep $R_3(\theta)$

$$R_3(\theta_2) \cdot R_3(\theta_1) = R_3(\theta_1+\theta_2)$$

$$\Rightarrow R_3(\theta) = \lim_{n\to\infty} \underbrace{R_3\left(\frac{\theta}{n}\right) \cdots\cdots R_3\left(\frac{\theta}{n}\right)}_{n}$$

$$= \lim_{n\to\infty} \left(\mathbb{1} - i\frac{\theta}{n} L_3 \right)^n \qquad e^x = \lim_{n\to\infty} \left(1+\frac{x}{n}\right)^n$$

$$= \exp\left(-i\theta L_3\right) = R_3(\theta) = \exp(\hat{\theta} L_3)$$

positive angle convention

physicist convention

generator

$$\hat{\theta} = -i\theta$$

G: $g(\alpha)$, $g(\beta)$

$$g(\alpha)\, g(\beta) = g(h(\alpha,\beta)) \qquad h(\alpha,\beta) \overset{?}{=} h(\beta,\alpha)$$

$$g(\beta)\, g(\alpha) = g(h(\beta,\alpha))$$

if $\alpha = 0$: $\quad h(0,\beta) = \beta$

$\qquad\qquad h(\beta, 0) = \beta \quad \checkmark$

$\beta = 0$: $\quad \checkmark$

$$g(\alpha) = \mathbb{1} + \alpha^I T_I + \alpha^I \alpha^J T_{IJ} + \cdots$$

$$g(\beta) = \mathbb{1} + \beta^I T_I + \beta^I \beta^J T_{IJ} + \cdots$$

$$g(h(\alpha,\beta)) = \mathbb{1} + h^I(\alpha,\beta)\, T_I + h^I h^J T_{IJ} + \cdots$$

$$h^I(\alpha,\beta) = \alpha^I + \beta^I + C^I{}_{JK}\,\alpha^J\beta^K + \cdots$$

plug back into $g(h(\alpha,\beta))$ expansion

compare $g(\alpha)\,g(\beta) = g(h(\alpha,\beta))$ in each order of α, β

$$\text{LHS} = \mathbb{1} + \beta^I T_I + \beta^I\beta^J T_{IJ} + \alpha^I T_I + \alpha^I T_I\,\beta^J T_J$$
$$+ \alpha^I\alpha^J T_{IJ}\,\beta^K T_K + \alpha^I T_I\,\beta^J\beta^K T_{JK} + \cdots$$

$$\text{RHS} = \mathbb{1} + \alpha^I T_I + \beta^I T_I + C^I{}_{JK}\,\alpha^J\beta^K T_I + \cdots$$

	LHS	RHS
no α, β	$\mathbb{1}$	$\mathbb{1}$
1st order α, β	$\alpha^I T_I + \beta^I T_I$	$\alpha^I T_I + \beta^I T_I$
2nd order α, β	$\beta^I \beta^J T_{IJ} + \alpha^I T_I \beta^J T_J$ $+ \alpha^I \alpha^J T_{IJ}$	$C^I{}_{JK} \alpha^J \beta^K T_I$ $+ \alpha^I \alpha^J T_{IJ}$ $+ \alpha^I \beta^J T_{IJ}$ $+ \beta^I \alpha^J T_{IJ}$ $+ \beta^I \beta^J T_{IJ}$

$$\Rightarrow \quad \alpha^I \beta^J T_I T_J = C^I{}_{JK}\, \alpha^J \beta^K T_I$$

$$+ \underbrace{\alpha^I \beta^J T_{IJ} + \beta^I \alpha^J T_{IJ}}$$

since $T_{IJ} = T_{JI}$:

$$\beta^I \alpha^J T_{IJ} = \alpha^I \beta^J T_{JI} = \alpha^I \beta^J T_{IJ}$$

$$\Rightarrow \quad 2\underbrace{\alpha^I \beta^J}\, T_{IJ} = \underbrace{\alpha^I \beta^J}\, T_I T_J + \underbrace{\alpha^I \beta^J}\, C^K{}_{IJ} T_K$$

$$\Rightarrow \quad T_{IJ} = \frac{1}{2}\left\{ T_I T_J + C^K{}_{IJ} T_K \right\}$$

$$\left.\begin{aligned}
&[T_I, T_J] = \ ? \\
&[[T_I, T_J], T_K] + \text{cyclic} = 0
\end{aligned}\right\} \quad \text{Homework.}$$

start from $T_{IJ} = T_{JI}$:

$$\frac{1}{2}\left\{ T_I T_J + c^K{}_{IJ} T_K \right\} = \frac{1}{2}\left\{ T_J T_I + c^K{}_{JI} T_K \right\}$$

$$\Rightarrow [T_I, T_J] = \left(c^K{}_{JI} - c^K{}_{IJ} \right) T_K$$

Jacobi identity:

$$[[T_I, T_J], T_K] + [[T_J, T_K], T_I] + [[T_K, T_I], T_J]$$

$$= \left[C^{L}{}_{[JI]} T_L , T_K \right] + C^{L}{}_{[KJ]} \left[T_L, T_I \right] + C^{L}{}_{[IK]} \left[T_L, T_J \right]$$

$$= \left[C^{L}{}_{[JI]} C^{M}{}_{[KL]} + C^{L}{}_{[KJ]} C^{M}{}_{[IL]} + C^{L}{}_{[IK]} C^{M}{}_{[JL]} \right] T_M$$

$$= 0$$

$$\Rightarrow C^{L}{}_{[IJ]} C^{M}{}_{[LK]} + C^{L}{}_{[JK]} C^{M}{}_{[LI]} + C^{L}{}_{[KI]} C^{M}{}_{[LJ]} = 0$$

Lecture 3: groups, spin, SU(3)

* finite group : $G = \{g_1, g_2, \dots, g_N\}$ $N \to \infty$ but has to be countable.

e.g. integer numbers $\mathbb{Z}$.

$$R(\alpha, \beta, \gamma) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha \\ 0 & \sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} \cos\beta & 0 & \sin\beta \\ 0 & 1 & 0 \\ -\sin\beta & 0 & \cos\beta \end{pmatrix} \begin{pmatrix} \cos\gamma & -\sin\gamma & 0 \\ \sin\gamma & \cos\gamma & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

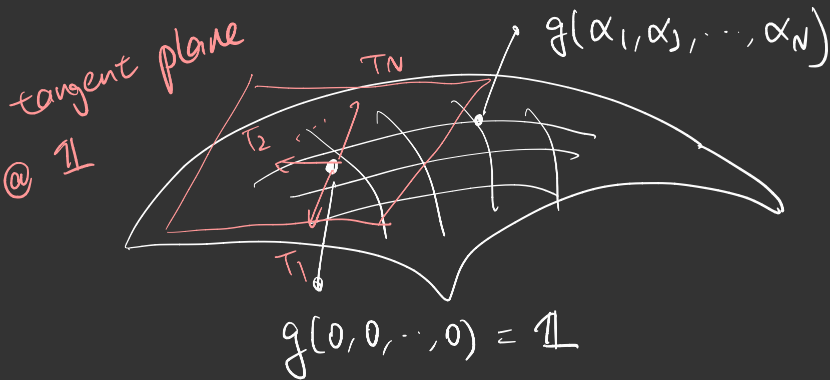
$$= \exp\left[-i(\tilde{\alpha} L_1 + \tilde{\beta} L_2 + \tilde{\gamma} L_3)\right]$$

different but equivalent representations.

$\{\tilde{\alpha}, \tilde{\beta}, \tilde{\gamma}\}$ & $\{\alpha, \beta, \gamma\}$ are related

[see Baker-Campbell-Hausdorff formula]

You can think of $\{\alpha, \beta, \gamma\}$ as "coordinate"



(clearly, $T_I \notin G$.
they are living in the tangent plane)

and we can ask what if I consider the group element near the identity. : expand around $\mathbb{1}$,

$$g(\alpha) = \mathbb{1} + \alpha^I T_I + \alpha^I \alpha^J T_{IJ} + \cdots$$

define the generator T_I.

Use closure

$$g(\alpha)\, g(\beta) = g(h(\alpha, \beta)) \Rightarrow [T_I, T_J] = f_{IJ}{}^{K}\, T_K$$

$$f_{IJ}{}^{K} = -f_{JI}{}^{K}$$

structure constant.

(in the homework, we have $f_{IJ}{}^{K} = C_{IJ}{}^{K} - C_{JI}{}^{K}$

Look back our rotation: $\{L_1, L_2, L_3\}$

$$[L_1, L_2] = iL_3$$

$$[L_I, L_J] = i\,\epsilon_{IJK}\, L_K$$

Levi-civita symbol

$$\epsilon_{IJK} = \begin{cases} 1 & (123) \\ -1 & (321) \\ 0 & \text{other} \end{cases}$$

$$g(\alpha) = \mathbb{1} + \alpha^I T_I^{(M)} + O(\alpha^2)$$ Mathematician

$$= \mathbb{1} - i\alpha^I (iT_I) + \cdots$$

Physician

$$T_I^{(P)} = i T_I^{(M)} \longrightarrow T_I^{(M)} = -i T_I^{(P)}$$

$$[T_I^{(M)}, T_J^{(M)}] = f_{IJ}{}^K T_K^{(M)}$$

Aside: Spin $|\vec{S}|^2 = j(j+1)\hbar^2$

j: spin

j: integer $\Rightarrow$ bosons : photon, higgs.

j: half-integer $\Rightarrow$ fermions : electron light : E+M , polarizations : $\begin{cases} \text{helicity } +1 & \text{left-handed polarization} \\ \phantom{\text{helicity }} -1 & \text{right-handed} \ \ldots \end{cases}$

"Maxwell Equation"

Beyond Standard Model:

$\hat{j} = \frac{3}{2}$, $\hat{j} = 2$

gravitino graviton

Associativity:

$$\big(g(\alpha)\, g(\beta)\big)\, g(\gamma) = g(\alpha)\, \big(g(\beta)\, g(\gamma)\big)$$

$$\Downarrow \quad g(\alpha) = \mathbb{1} - i\alpha^I T_I \,,\; g(\beta),\; g(\gamma) = \cdots$$

Jacobi Identity (for any Lie group, its generators satisfy this)

$$[[T_I, T_J], T_K] + [[T_J, T_K], T_I] + [[T_K, T_I], T_J] = 0$$

(check by yourself:)

Lie Algebra :

1) T_I , $I = 1, \cdots, N$

generators of a Lie group G.

2) $[T_I, T_J] = i f_{IJ}{}^{K} \, T_K$

3) Jacobi identity

$$[\underbrace{[T_i, T_j]}_{= i f_{ij}{}^{l} T_l}, T_k] + \text{cyclic} = 0$$

"constant"

※ $[c_1 A + c_2, B] = c_1 [A, B]$

$$\Rightarrow \quad i f_{ij}{}^{l} [T_l, T_k] + \cdots = 0$$

$$\Rightarrow \quad i f_{ij}{}^{l} \, i f_{lk}{}^{p} \, T_p + i f_{jk}{}^{l} \, i f_{li}{}^{p} \, T_p + i f_{ki}{}^{l} \, i f_{lj}{}^{p} \, T_p = 0$$

$$\Rightarrow \quad \left(f_{ij}{}^{l} f_{lk}{}^{p} + f_{jk}{}^{l} f_{li}{}^{p} + f_{ki}{}^{l} f_{lj}{}^{p} \right) T_p = 0$$

$$(\quad)^{p} \, T_p = 0$$

since T_p's linearly independent.

$(\quad)^{p} = 0$

$$f_{ij}{}^{l} f_{lk}{}^{p} + f_{jk}{}^{l} \underbrace{f_{li}{}^{p}}_{= -f_{il}{}^{p}} + \underbrace{f_{ki}{}^{l}}_{-f_{ik}{}^{l}} \underbrace{f_{lj}{}^{p}}_{-f_{jl}{}^{p}} = 0$$

$$\Rightarrow (-i) f_{ik}{}^{l} (-i) f_{jl}{}^{p} - (-i) f_{jk}{}^{l} (-i) f_{il}{}^{p} = + i f_{ij}{}^{l} (-i) f_{lk}{}^{p}$$

let $(A_i)_k{}^l \equiv -i f_{ik}{}^{l}$

$$\Rightarrow [A_i, A_j] = i f_{ij}{}^{l} A_l$$

this means is a representation called "Adjoint representation".

Homework: find the Adjoint rep for rotation $SO(3)$

3x3

special orthogonal

generators: $\{L_1, L_2, L_3\}$

$$[L_i, L_j] = i\epsilon_{ijk} L_k$$

answer:

$$t_1 = \begin{pmatrix} & & \\ & & \\ & & \end{pmatrix} \qquad t_2 = \begin{pmatrix} & & \\ & & \\ & & \end{pmatrix}$$

$$t_3 = \begin{pmatrix} & & \\ & & \\ & & \end{pmatrix}$$

$$(t_i)_k{}^l = -i f_{ik}{}^l \qquad i, k, l = 1, 2, 3$$

$$= -i\,\epsilon_{ik}{}^l \qquad (\text{for now, } \epsilon_{ik}{}^l = \epsilon_{ikl})$$

$$[t_i, t_j] = i f_{ij}{}^{k}\, t_k \qquad (*)$$

1) take the $*$ of $(*)$ complex

$$[\,\_\,,\,\_\,] = i f_{ij}{}^{k}\, \_$$

2) T (traspose) dual

3) $\dagger$ (hermitian conj.) complex dual

SU(3) = the group of 3×3 unitary matrices with det = 1

SO(3) : the group of 3×3 orthogonal matrices with det = 1

orthogonal

special : det = 1

O, O^T : $OO^T = O^TO = \mathbb{1}$

$O^T = O^{-1}$

unitary : U, $U^\dagger$, $UU^\dagger = U^\dagger U = \mathbb{1}$

$U^\dagger = (U^*)^T$

$X_a = X_a^\dagger$

SU(3) : generated by 3×3 hermitian, traceless matrices. X_a

$g \in SU(3)$ $\quad g = \exp(i\alpha_a X_a)$ $\quad (\det(g) = 1)$

$$X = \begin{pmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{pmatrix}$$

$$X = X^{\dagger} \Rightarrow \begin{cases} a_4 = a_2^* \\ a_7 = a_3^* \\ a_8 = a_6^* \\ a_1 = a_1^* \\ a_5 = a_5^* \\ a_9 = a_9^* \end{cases}$$

$3 \times 2 + 3 - 1 = 8$

3 complex

a_2, a_3, a_6

3 real

a_1, a_5, a_9

traceless

$a_1 + a_5 + a_9 = 0$

choose a basis for 3x3 traceless hermitian matrices.

Gell - Mann matrices (generalizations of Pauli matrices)

$\sigma_1, \sigma_2, \sigma_3$

$$\lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \lambda_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \quad \lambda_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}$$

$$\lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \quad \lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix} \quad \lambda_8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

$$\mathrm{Tr}(\lambda_a \lambda_b) = 2\delta_{ab} \qquad \delta_{ab} = \begin{cases} 0 & a \neq b \\ 1 & a = b \end{cases}$$ Kronecker delta

"dimension" = # of generators. $\quad SU(N)$: $\dim = N^2 - 1$

By convention, we define generators

$$X_a = \frac{1}{2}\lambda_a \quad \text{for } SU(3) \qquad \mathrm{Tr}(X_a X_b) = \frac{1}{2}\delta_{ab}$$

$$[X_a, X_b] = i f_{ab}{}^{c} X_c$$

The X_a's provide a <u>3</u>-dim'l rep of $SU(\underline{3})$ Lie algebra

this is called the defining representation (fundamental)

Group work:

$$[X_a, X_b] = i f_{ab}{}^{c} X_c$$

$$a,b = \underline{1,2} \qquad c = 1, \cdots, 8$$

$$1,3$$

$$1,4$$

$$f_{123} = 1, \qquad f_{12c} = 0 \quad \text{for } c \neq 3$$

$$f_{132} = -1, \qquad f_{13c} = 0 \quad (c \neq 2)$$

$$f_{147} = \frac{1}{2}, \qquad f_{14c} = 0 \quad c \neq 7$$

$X_3 = \frac{1}{2}\lambda_3, \quad X_8 = \frac{1}{2}\lambda_8$ commute $[X_3, X_8] = 0$

since λ_3, λ_8 are diagonal.

$\Rightarrow \{X_3, X_8\}$: Cartan subalgebra

# of generators in the Cartan subalgebra = rank.

※ $\{\sigma_1, \sigma_2, \sigma_3\}$ generators of $SU(2)$ rank = 1.

$S_i = \frac{1}{2}\sigma_i \qquad S_3 = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}$

try: $\mathrm{Tr}(S_i S_j) = ?$

$S_3 \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad S_3 \begin{pmatrix} 0 \\ 1 \end{pmatrix} = -\frac{1}{2}\begin{pmatrix} 0 \\ 1 \end{pmatrix}$

For $SU(3)$:

$$H_1 \equiv X_3 = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad H_2 \equiv X_8 = \frac{1}{2\sqrt{3}}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

states: eigenvector

$$|\vec{\mu}\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

eigenvalues:

$$\vec{\mu} = \left(\frac{1}{2}, \frac{\sqrt{3}}{6} \right)$$

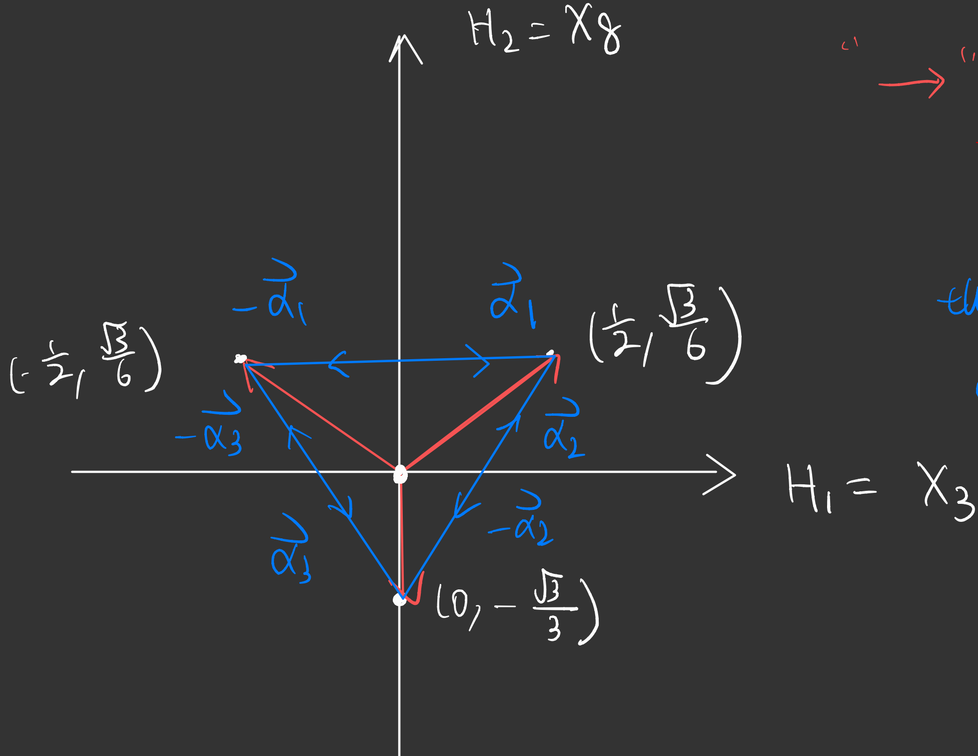
"weight vectors"

eigenvalue of X_3

eigenvalue of X_8

$$\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} : \left(-\frac{1}{2}, \frac{\sqrt{3}}{6} \right)$$

$$\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} : \left(0, -\frac{1}{\sqrt{3}} \right)$$



$\longrightarrow$ " weight vectors for fundamental reps of SU(3)

the root vectors are the differences between various pairs of weight vectors μ (in any rep)

Adjoint reps: $[X_a, X_b] = i f_{abc} X_c$

$(t_a)_{bc} = -i f_{abc}$ we know 8 8×8 matrices

Example: $H_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ $f_{ijk} = 0$

$H_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ (abelian group)

states	Eigenvectors:	Eigenvalues H_1	H_2	weight vector $\vec{\mu}$
$\|1,1\rangle \equiv \begin{pmatrix} 1 \\ 0 \end{pmatrix}$		$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = 1\begin{pmatrix} 1 \\ 0 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = 1\begin{pmatrix} 1 \\ 0 \end{pmatrix}$	$(1,1)$
$\|1,-1\rangle \equiv \begin{pmatrix} 0 \\ 1 \end{pmatrix}$		$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = 1\begin{pmatrix} 0 \\ 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = -1\begin{pmatrix} 0 \\ 1 \end{pmatrix}$	$(1,-1)$

$$\mathrm{Tr}(T_a T_b) = \frac{1}{2}\delta_{ab}$$

t_3, t_8 : simultaneously diagonalize them

$\Rightarrow$ diagonal entry pairs (t_3, t_8) (8 of them)

$\Rightarrow$ weight vectors for adjoint reps. = "root vectors"

$SU(N)$: dim $= N^2 - 1$ = dim (adjoint rep)

rank $= N - 1$

$N^2 - 1 - (N-1)$

Lecture 4: symmetry

Quick Summary of Lie algebra

Given Hermitian generators X_a satisfying $[X_a, X_b] = i f_{abc} X_c$

[choose normalization as $\mathrm{Tr}(X_a X_b) = k\,\delta_{ab}$]

Step 1: find a maximal set of simultanously commuting generators

these generate the Cartan subalgebra

$\{H_1, H_2, \cdots, H_m\}$ $\quad$ m= rank of the algebra

$$H_i^\dagger = H_i \;;\; [H_i, H_j] = 0 \;;\quad \mathrm{Tr}[H_i H_j] = k\,\delta_{ij}$$

Step 2: use a basis of states where each H_i is diagonal (for any representation)

so that $H_i |\mu, x\rangle = \underbrace{\mu_i}_{\text{weight}} |\mu, x\rangle$

$\mu = (\mu_1, \mu_2, \cdots, \mu_m)$: weight vector

x = other labels, to distinguish these various states with same weight vector.

Step 3: turn our attention to adjoint rep. (N, $N \times N$ matrices)

basis of states: $|t_1\rangle, |t_2\rangle, \cdots |t_N\rangle$ N = dim of the algebra.

$H_i |t_a\rangle = \alpha_i |t_a\rangle$ $i = 1, \cdots, m$

root vector $\alpha_a = (\alpha_1, \cdots, \alpha_m)$ $a = 1, \cdots, N$

Note: "ket"

$$t_a |t_b\rangle = \sum_c |t_c\rangle \underbrace{\langle t_c| t_a |t_b\rangle}$$

$(N^2-1)\times(N^2-1)$ matrix rep of X_a

$$= \sum_c |t_c\rangle \underbrace{[t_a]_{cb}}$$

$$= \sum_c (-i f_{acb}) |t_c\rangle = \sum_c i f_{abc} |t_c\rangle$$

$$= | i f_{abc} t_c \rangle = | [t_a, t_b] \rangle$$

f_{abc} is fully antisymmetric ϵ_{ijk}

$f_{abc} = -f_{bac}$
$= -f_{acb}$

$$\Rightarrow \boxed{t_a |t_b\rangle = |[t_a, t_b]\rangle \text{ for adjoint rep } t_a}$$

※ $H_i |H_j\rangle = |[H_i, H_j]\rangle = 0$

$\Rightarrow$ we would have m zero root vectors.

SU(2)

$|t_1\rangle = \begin{pmatrix}1\\0\end{pmatrix}$, $|t_2\rangle = \begin{pmatrix}0\\1\end{pmatrix}$ 2

"bra"

$$\mathbb{1} = \sum_{i=1} |t_i\rangle\langle t_i|$$

$$= \begin{pmatrix}1\\0\end{pmatrix}(1\ \ 0) + \begin{pmatrix}0\\1\end{pmatrix}(0\ \ 1) = \begin{pmatrix}1&0\\0&1\end{pmatrix}$$

$$\langle t_1| = (|t_1\rangle)^T = (1\ \ 0)$$

$$A = \begin{pmatrix}a&b\\c&d\end{pmatrix}$$

$$\langle t_1|A|t_2\rangle = (1\ \ 0)\begin{pmatrix}a&b\\c&d\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix} = b = (A)_{12}$$

Step 4: look at nonzero root vectors, I would like to call these states that are not corresponding to H's as E_α. $\Rightarrow$ $H_i|E_\alpha\rangle = \alpha_i|E_\alpha\rangle$ $(\alpha_i \neq 0, \alpha_i \in \mathbb{R})$

N^2-1: X_a's $\begin{cases} H_i & i=1,\dots,m \\ E_\alpha \end{cases}$

$\Rightarrow |[H_i, E_\alpha]\rangle = \alpha_i|E_\alpha\rangle \Rightarrow [H_i, E_\alpha] = \alpha_i E_\alpha$

※ these E_α's are some linear combination of t_a's.

they're not hermitian: $[H_i, E_\alpha]^\dagger = -[H_i^\dagger, E_\alpha^\dagger]$

$= \alpha_i E_\alpha^\dagger$

Let's call $E_\alpha^\dagger = E_{-\alpha}$ since $[H_i, E_\alpha^\dagger] = -\alpha_i E_\alpha^\dagger$

$\Rightarrow$ root vectors come in pairs $\alpha, -\alpha$

Consider E_α, $E_{-\alpha}$ as raising / lowering operators.

e.g. for any state $|\mu\rangle$ (in any rep)

$$H_i|\mu\rangle = \mu_i|\mu\rangle$$

look at $$H_i E_{\pm\alpha}|\mu\rangle = [H_i, E_{\pm\alpha}]|\mu\rangle + E_{\pm\alpha} H_i|\mu\rangle$$
$$= \pm\alpha E_{\pm\alpha}|\mu\rangle + E_{\pm\alpha}\mu_i|\mu\rangle$$
$$= (\mu_i \pm \alpha) E_{\pm\alpha}|\mu\rangle$$

$\Rightarrow$ weight of $E_{\pm\alpha}|\mu\rangle$ are $\mu \pm \alpha$.

Step 5: consider $E_\alpha |E_{-\alpha}\rangle$: weight $= \alpha - \alpha = 0$

$$E_\alpha |E_{-\alpha}\rangle = \sum_{i=1}^{m} \beta_i |H_i\rangle$$

using orthogonality condition, $\beta_i = \alpha_i$

$\Rightarrow [E_\alpha, E_{-\alpha}] = \alpha \cdot H$ true for any rep!

$\Rightarrow$ form a SU(2) subalgebra!

To be more precise,

$$\begin{cases} E_\alpha^{\pm} = \frac{1}{|\alpha|} E_{\pm\alpha} \\ E_\alpha^{3} = \frac{1}{|\alpha|^2} \alpha \cdot H \end{cases}$$

$$[E_\alpha^3, E_\alpha^{\pm}] = \pm E_\alpha^{\pm}$$

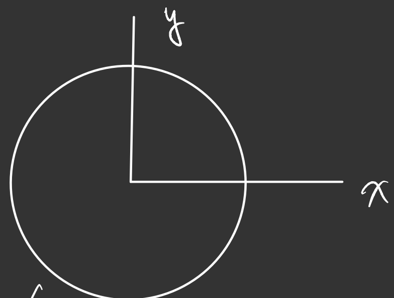
$$[E_\alpha^+, E_\alpha^-] = E_\alpha^3$$

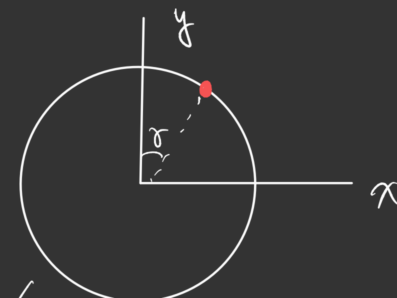
(same for J^+, J^-, J^3 in SU(2))

Symmetry: an operator that when acting on an object leaves something unchanged.

Invariant: a quantity that is left unchanged.

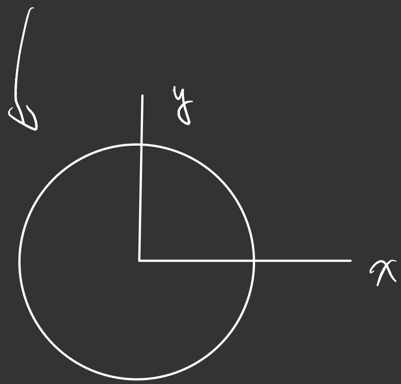
e.g. circle





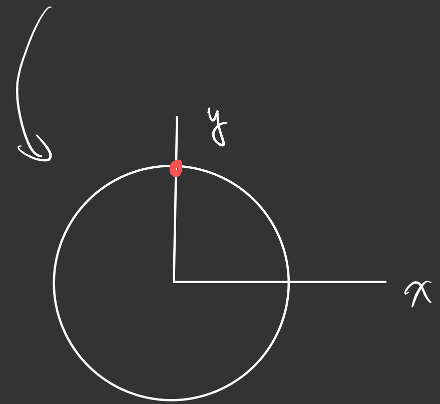
$e^{i\gamma L_3}$

$e^{i\gamma L_3}$

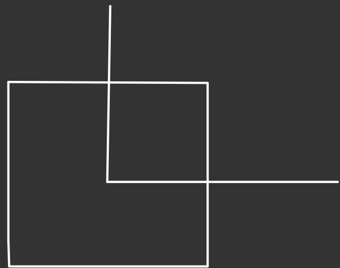


(you can't tell the circle is rotating w/o any mark)

$\Rightarrow$ symmetry ✓

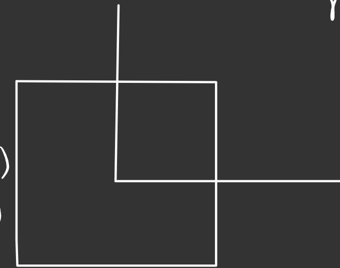


Not a symmetry !



?

C_1 (flip around x)
C_2 (- - - y)
C_3 ($y = x$)
C_4 ($y = -x$)



rotation

$R_1 = R(\frac{\pi}{2})$ $\quad R_2 = R(\pi)$

$R_3 = R(\frac{3\pi}{2})$ $\quad R_4 = R(2\pi)$

they actually form a group $\{C_1, C_2, C_3, C_4, R_1, R_2, R_3, R_4\}$

Invariants: area, angle, perimeter

Rotational Symmetry

quantity $\mathcal{L}(x, y, z)$ that is "invariant" under a rotation

$$\mathcal{L}(x', y', z') = \mathcal{L}(x, y, z)$$

$$C_0 = R(\alpha, \beta, \gamma)\, \mathcal{L}(x, y, z)$$

consider infinitesimal angles α, β, γ. LHS can be expanded

since $R(\alpha, \beta, \gamma) = \exp[-i(\alpha L_1 + \beta L_2 + \gamma L_3)]$

$$\Rightarrow \mathcal{L}(x', y', z') = [1 - i(\alpha L_1 + \beta L_2 + \gamma L_3)] \mathcal{L}(x, y, z)$$

$$= \mathcal{L}(x, y, z)$$

$$\Rightarrow \delta_{R(\alpha^i)} \mathcal{L} = \mathcal{L}(x, y, z) - \mathcal{L}(x', y', z')$$

$$= i(\alpha L_1 + \beta L_2 + \gamma L_3) \mathcal{L}(x, y, z) = 0$$

since α, β, γ are independent : $\begin{cases} L_1 \mathcal{L} = 0 \\ L_2 \mathcal{L} = 0 \\ L_3 \mathcal{L} = 0 \end{cases} \Leftrightarrow L_i \mathcal{L} = 0$

If any quantity only satisfies $L_3 \mathcal{L} = 0$: $\mathcal{L}$ possess rotational sym about z-axis.

Consider there is some system that possesses a sym about z-axis.

and has an energy $\mathcal{E}_t(x, y)$

measurement: $\mathcal{E}_t(x, y=0) = A_0 x^4 \equiv \mathcal{E}_{SM}(x)$

$$\Rightarrow \mathcal{E}_t(x, y) = \mathcal{E}_{SM}(x) + \mathcal{E}_{SmSM}(x, y)$$

where $\mathcal{E}_{SmSM}(x, 0) = 0$

(symmetry - modified standard measurement)

$$L_3 \mathcal{E}_t = 0 \implies 0 = L_3 \left[\mathcal{E}_{SM}(x) + \mathcal{E}_{SMSM}(x, y) \right]$$

$$= i 4 A_0 x^3 y + L_3 \left[\mathcal{E}_{SMSM}(x, y) \right]$$

set $\mathcal{E}_{SMSM}(x, y) = \sum_{n=1}^{\infty} y^n f_n(x)$

$$\implies 0 = 4 A_0 x^3 y + \sum_{n=1}^{\infty} \left[y^{n+1} \frac{df_n}{dx} - n y^{n-1} x f_n(x) \right]$$

$$= -x f_1(x) + \left[4 A_0 x^3 - 2 x f_2(x) \right] y$$

$$+ \sum_{n=1}^{\infty} y^{n+1} \left[\frac{df_n}{dx} - (n+2) x f_{n+2}(x) \right]$$

at each level of $y = 0$

$\Rightarrow \quad y^0: \quad x f_1(X) = 0 \qquad f_1(X) = 0$

$y^1: \quad 4A_0 x^3 - 2x f_2(X) = 0 \quad \rightarrow \quad f_2(X) = 2A_0 x^2$

$y^{n+1} \ (n \geq 1): \quad f_{n+2}(X) = \dfrac{1}{(n+2)} \dfrac{1}{x} \dfrac{df_n}{dx}$

$f_2 = 2A_0 x^2, \quad f_4 = \frac{1}{4x} \cdot 4A_0 X = A_0$

all others $= 0$

$\Rightarrow \quad \mathcal{E}_t = A_0 x^4 + 2A_0 x^2 y^2 + A_0 y^4$

$= A_0 (x^2 + y^2)^2$

suppose x^4 was instead $\cos(x)$, guess $\mathcal{E}_t = \cos\sqrt{x^2+y^2}$

suppose initial function was $A_0 x \Rightarrow \mathcal{E}_t = A_0 \sqrt{x^2+y^2}$

$\log x \Rightarrow \mathcal{E}_t = A_0 \log \sqrt{x^2+y^2}$

$A_0 x^{137} \Rightarrow \mathcal{E}_t = A_0 (x^2+y^2)^{137/2}$