Celestial Quantum Error Correction:

From Non-commutative Geometry to Celestial CFT

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Based on 2312.16298 & wip with Alfredo Guevara

The Question

The connection between quantum information and holography has been studied extensively, primarily through the AdS/CFT correspondence.

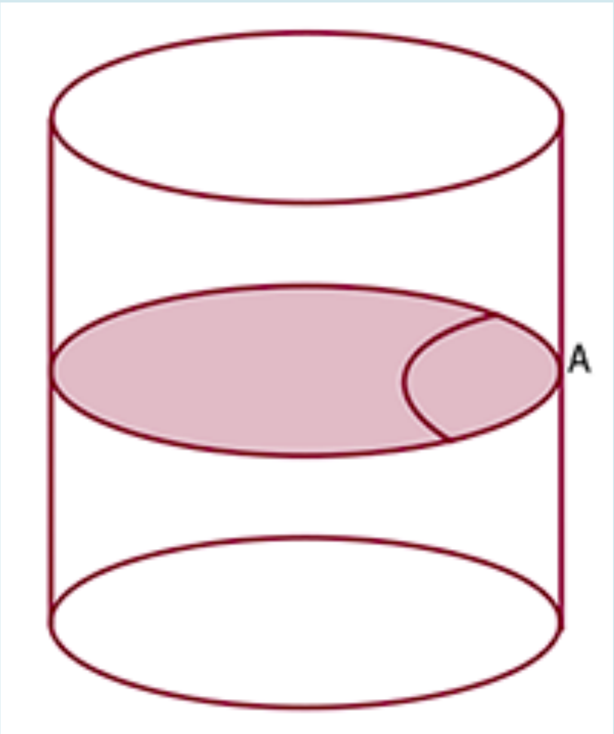
Can we extend analogous concepts to the celestial holography?

Prelude: Geometry & Entanglement

AdS/CFT Correspondence

- Ryu-Takayanagi (RT) formula

$$S_A = \frac{\text{area of } \gamma_A}{4G}$$

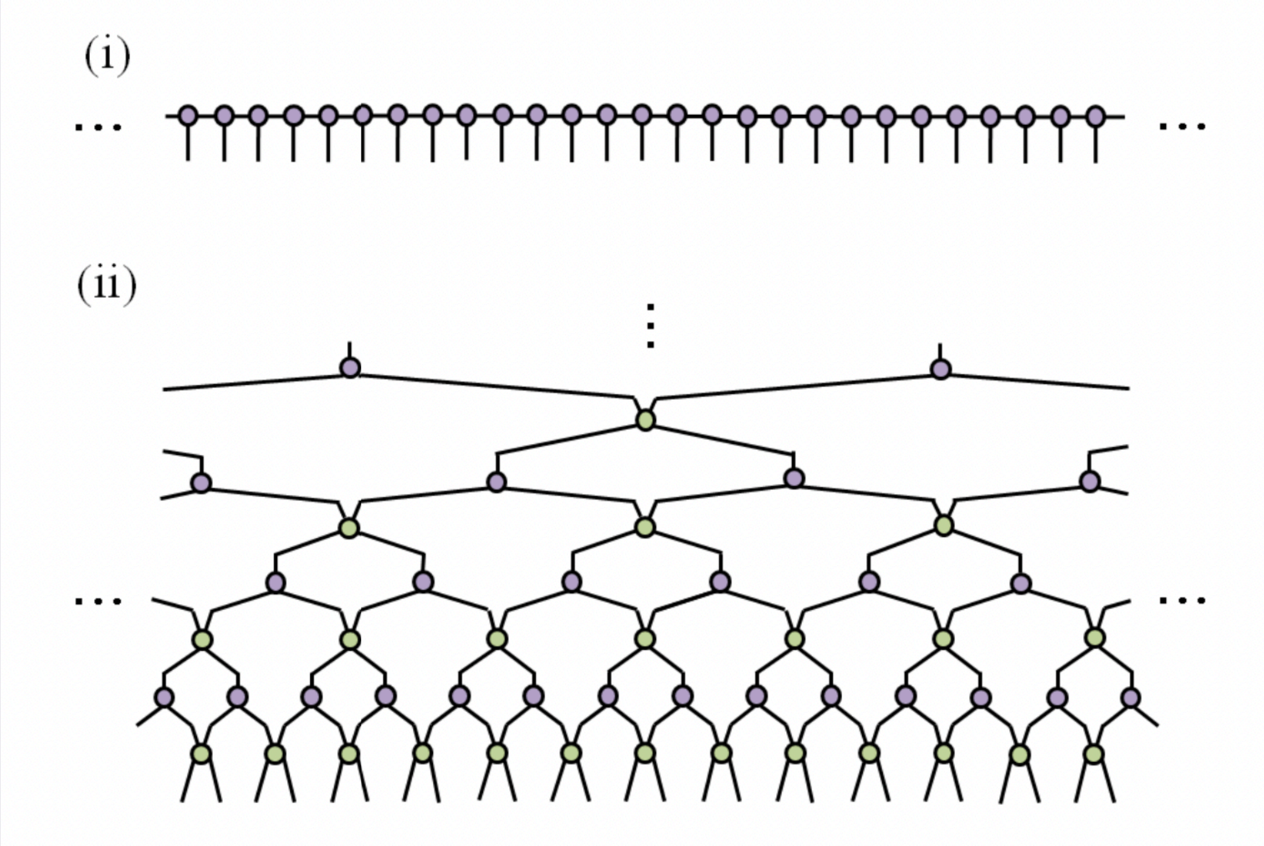


[Ryu, Takayanagi, '06]

Condensed Matter Physics

- Tensor Networks: represent quantum many-body states

- Different types of tensor network states generate different geometries



[Swingle, '09]

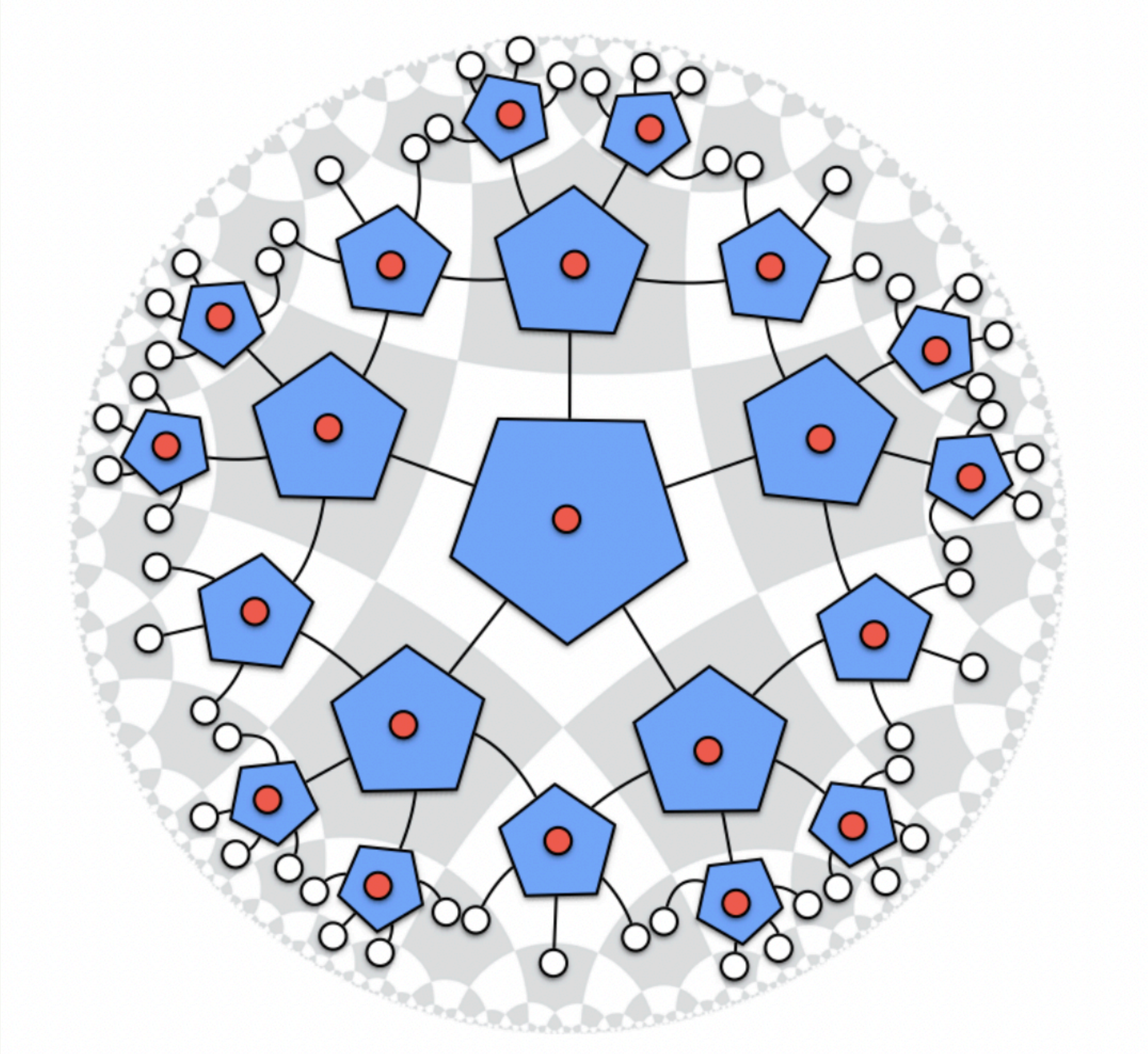
[Evenbly, Vidal, '11]

Prelude: holographic code

- Lesson learned from Swingle [Swingle, '09]: some physics of AdS/CFT can be modeled by a MERA-like tensor network!

- [Almheiri, Dong, Harlow, '14] argued: the emergence of bulk locality in AdS/CFT can be characterized in terms of quantum error-correcting codes

- [Pastawski, Yoshida, Harlow, Preskill, '15] proposed such a toy model based on a tensor network construction of QECC

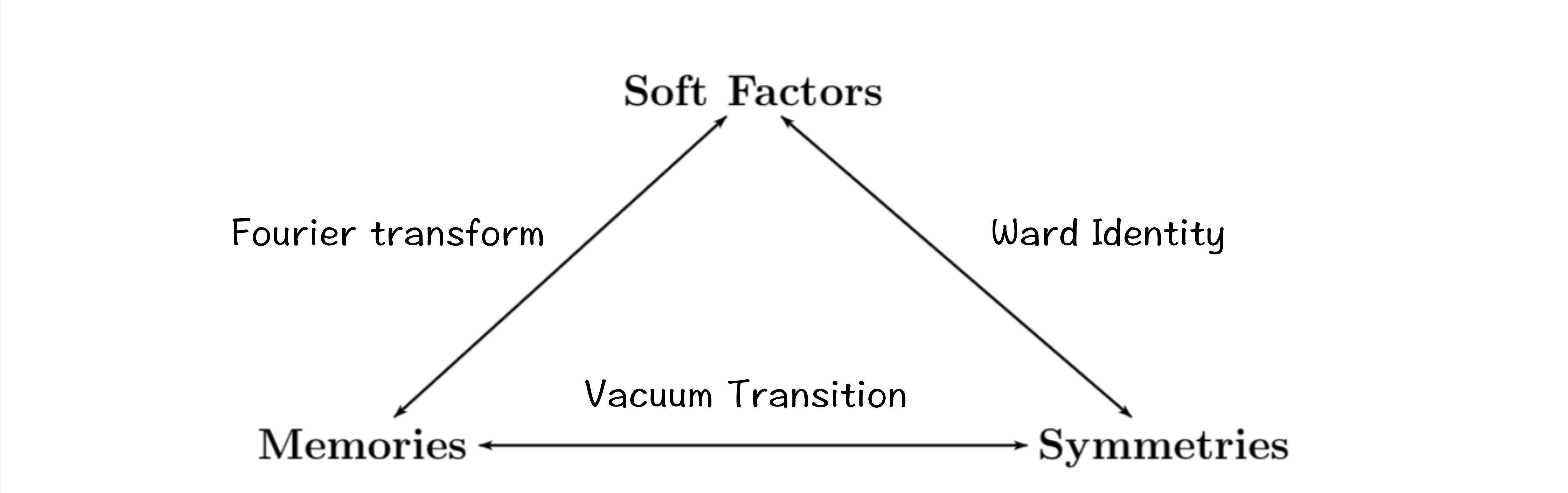


$$V : \mathscr{H}_{bulk} \to \mathscr{H}_{boundary} \quad , \quad |V|\psi\rangle| \approx ||\psi\rangle|$$

How about flat holography?

(Celestial)

- BMS symmetry spontaneous breaking: vacuum degeneracy
- IR divergence, Soft graviton & Goldstone modes



- $w_{1+\infty}$ hierarchy of soft currents

[He, Lysov, Mitra, Strominger, '14]

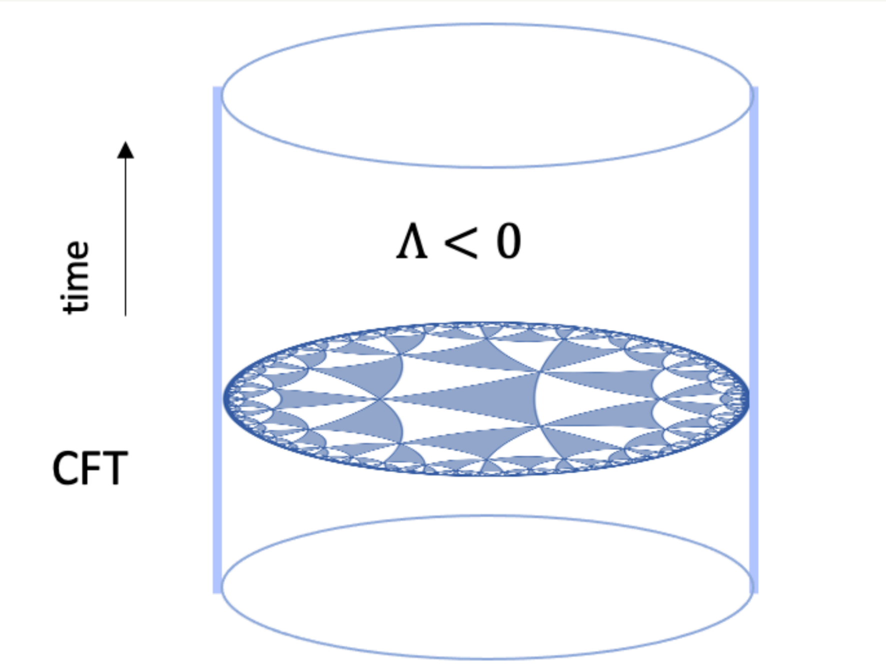
[Strominger, 1703.05448]

[Himwich, Narayanan, Pate, Paul, Strominger, '20]

[Guevara, Himwich, Pate, Strominger, '21]

[Strominger, '21], [Himwich, Pate, Singh '21]

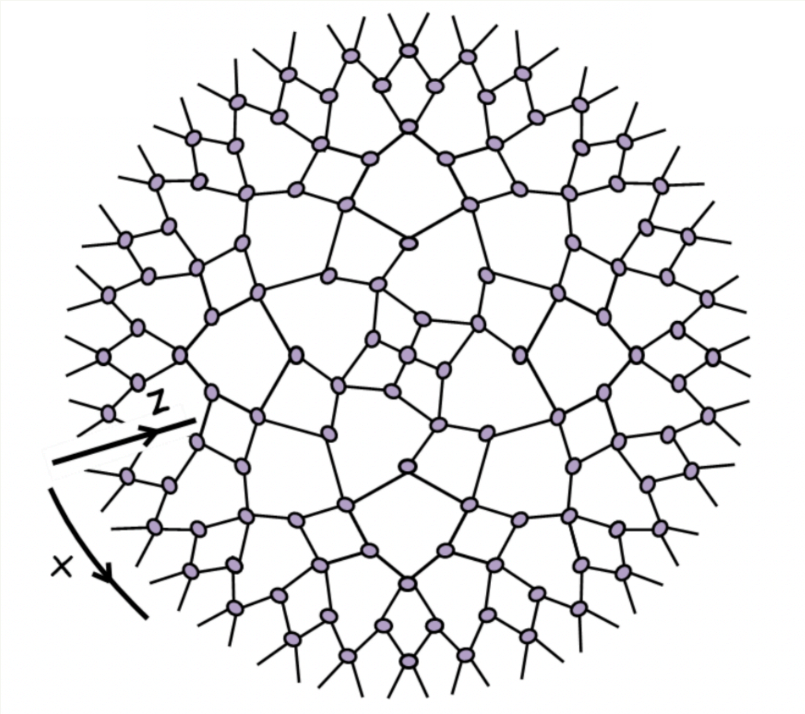
What we can do?



AdS/CFT Correspondence

Geometry & Entanglement

MERA-like Tensor Network

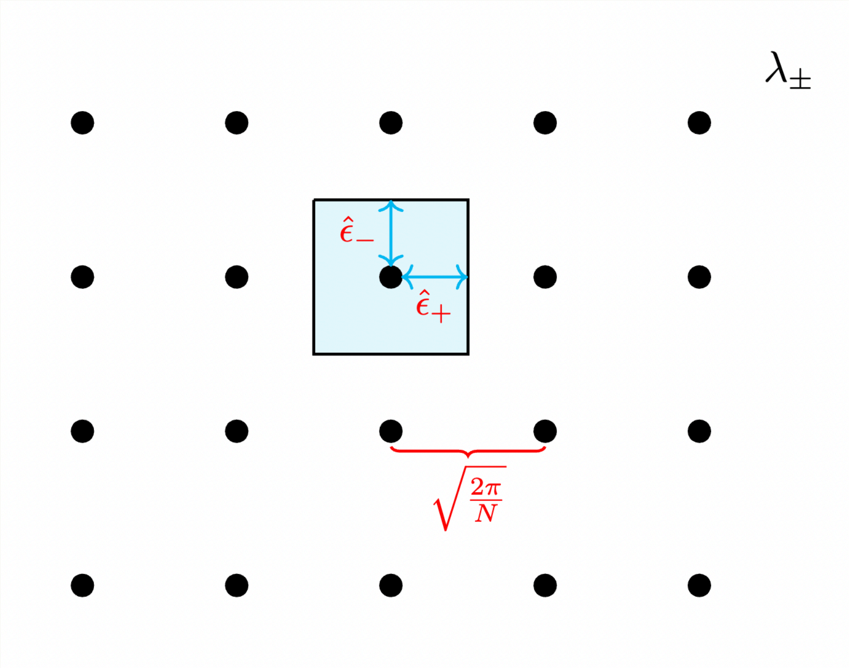


Celestial Holography:
$w_{1+\infty}$ hierarchy of soft currents
& its realization in twistor space

Algebraic structure &
Symmetries

Gottesman-Kitaev-Preskill Code

[Gottesman, Kitaev, Preskill, '00]



What we can do?

To unfold the implications of error correction in celestial holography, we would like to

- Understand the algebraic structure of GKP vs noncommutative geometry
- Guided by symmetries, construct a toy model of celestial CFT from QECC
- Show how the IR data is factored out whereas the hard data is protected

Algebraic structure $\longrightarrow$ Code $\longrightarrow$ Symmetries

Outline

Toy model with finite d.o.f.

From Fuzzy Spacetimes to Qubits

“n”: N-dimensional Hilbert space

From Qubits to Qunits

Celestial CFT from Quantum Error Correction

Emerge as $N \to \infty$

Outline

Toy model with finite d.o.f.

From Fuzzy Spacetimes to Qubits

From Qubits to Qunits

Celestial CFT from Quantum Error Correction

Noncommutative Klein Space

- Consider $\{x_i\}$ in $(2,2)$ signature $\mathbb{K}^{2,2}$

$$ds^2 \;=\; dz_1\, d\bar{z}_1 + dz_2\, d\bar{z}_2 \qquad z_1, \bar{z}_1 = x_1 \pm x_3 \;\;,\;\; z_2, \bar{z}_2 = x_2 \pm x_4$$

Real independent variables

- Radial distance

$$R^2 \;=\; z_1\bar{z}_1 + \bar{z}_2 z_2 \;=\; \det(x_{\alpha\dot{\alpha}}) \;\;,\;\; x_{\alpha\dot{\alpha}} \;=\; \begin{pmatrix} z_1 & -\bar{z}_2 \\ z_2 & \bar{z}_1 \end{pmatrix}$$

- Lorentz group $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})/\mathbb{Z}_2$ acts by

Null infinity: $|R^2| \to \infty$

$$x_{\alpha\dot{\alpha}} \;\to\; \Lambda_\alpha{}^\beta\, x_{\beta\dot{\beta}}\, \tilde{\Lambda}^{\dot{\beta}}{}_{\dot{\alpha}}$$

Noncommutative Klein Space

- Noncommutative structure: promote x_i to Hermitian operators satisfying the Wick algebra

$$[x_1, x_3] = [x_2, x_4] = \frac{i\tau}{2}$$

τ is a central term

$$\Rightarrow \quad [z_1, \bar{z}_1] = [z_2, \bar{z}_2] = -i\tau$$

- Euclidean spacetime: $z_1, \bar{z}_1 = x_1 \pm i x_3$ are not Hermitian but conjugate to each other
  - x_1, x_3 play roles of X, P while $z_1, \bar{z}_1$ are like $a, a^\dagger$
- Klein spacetime: $z_1, \bar{z}_1$ are Hermtian and obtained by canonical transf from x_1, x_3

The canonical symmetries contain the Lorentz group in $(2,2)$ signature!

Renormalization = Radial Direction

- The notion of holographic code emerges when we interpret τ as a renormalization scale
- Follows from noncommutative geometry: the noncommutative 3-sphere is constructed by introducing (in Euclidean signature)

$$\xi_i = R^{-1} z_i \quad , \quad \bar{\xi}_i = \bar{z}_i R^{-1}$$

τ is renormalized according to a radial scale

- Renormalized algebra

$$\left[\xi_i, \bar{\xi}_j\right] = -i\tilde{\tau}\,\delta_{ij} + O(1/R^4) \quad , \quad \tilde{\tau} = \tau R^{-2}$$

- Asymptotically
  - expect large R^2
  - null infty: $R^2 \to \infty$

Suggests the existence of the code in AFS

[Omori, Maeda, Miyazaki, Yoshioka, '98]

Twistor Embedding

- In flat space, twistor variables $(\lambda, \mu) \in \mathbb{RP}^3$ are defined projectively by

$$\text{incidence relation:} \quad \mu_\alpha(\lambda) = x_{\alpha\dot{\alpha}} \lambda^{\dot{\alpha}}$$

Projective property
$(\lambda, \mu) \sim t\,(\lambda, \mu)\ ,\ t \in \mathbb{R}$

- $\lambda^{\dot{\alpha}}$ can be parameterized by a homogeneous real coordinate z

$x^+ = \phi - \psi$ and (ϕ, ψ) are coordinates of the celestial torus

$$\lambda^{\dot{\alpha}} = \left(\sin\frac{x^+}{2}, \cos\frac{x^+}{2}\right) \sim \left(1, \frac{1}{z}\right) \quad , \quad z := \tan\frac{x^+}{2}$$

- Key observation:

$$\text{incidence relation} \rightarrow \quad \mu_\alpha(z) = \sum_{k=-1/2}^{1/2} \frac{\mu_\alpha^{(k)}}{z^{k+1/2}}$$

The global sector of its mode expansion

We can interpret $\mu_\alpha(z)$ as a conformal field with chiral weight $h = \dfrac{1}{2}$

Twistor Embedding

- $$\mu_\alpha(z) \;=\; \sum_{k=-1/2}^{1/2} \frac{\mu_\alpha^{(k)}}{z^{k+1/2}} \qquad\qquad \begin{aligned} \mu_+^{+1/2} &= -\,\bar{z}_2 \;\;, \;\; \mu_+^{-1/2} \;=\; z_1 \;\;, \\ \mu_-^{+1/2} &= \;\bar{z}_1 \;\;, \;\; \mu_-^{-1/2} \;=\; z_2 \;\;. \end{aligned}$$

Commutation relation: $\left[\mu_\alpha^{(k)}, \mu_\beta^{(j)}\right] \;=\; i\tau\,\delta^{k+j}\,\epsilon_{\alpha\beta} \qquad k, j = \pm\dfrac{1}{2}$

- Another note: the $\mathbb{RP}^1$ z-line can be continued to a circle S^1

$$\tilde{\mu}_\alpha(x^+) \;=\; \sum_{k=\pm 1/2} \tilde{\mu}_\alpha^{(k)} e^{ikx^+}$$

[Adamo, Mason, Sharma, '21], [Mason, '22]

From Fuzzy Spacetimes to Qubits

$\{\mu_+^{-1/2}, \mu_-^{+1/2}, \tau\}$ and $\{\mu_+^{+1/2}, \mu_-^{-1/2}, \tau\}$ are two reps of Heisenberg algebra

Commutation relation: $$\left[\mu_\alpha^{(k)}, \mu_\beta^{(j)}\right] = i\tau\,\delta^{k+j}\,\epsilon_{\alpha\beta} \qquad k, j = \pm\frac{1}{2}$$

- Recall: z and $\bar{z}$ play the role of X and P $\rightarrow$ natural to quantize the system in z, $\bar{z}$ or equivalently $\mu_\alpha^{(k)}$ $\rightarrow$ a pair of quantum harmonic oscillators
- Wick algebra only admits infinite-dimensional reps
- We need: finite-dimensional reps

From Fuzzy Spacetimes to Qubits

- Step 1: "exponentiate" the alegbra: Heisenberg lie algebra → Heisenberg lie group

$$g_{\pm}^{(k)} = e^{i\mu_{\pm}^{(k)}} \qquad \text{satisfying} \qquad \boxed{g_{+}^{(k)}\, g_{-}^{(-k)} = e^{i\tau}\, g_{-}^{(-k)}\, g_{+}^{(k)}}$$

Its rep acts on the Hilbert space

- Step 2: truncate the Hilbert space of a quantum harmonic oscillator ↦ the one of a qubit

For each k, the Heisenberg group realizes $U(2)$ if we take $\tau = \pi$!

General statement: $\tau = 2\pi/N$ realizes $U(N)$ for even N.

→ 2-qubit System

(Without loss of generality, we omit the k index and focus on one copy of them)

[Hoppe, '89]

2-qubit System

$$g_+ g_- = - g_- g_+$$

- Generators of $U(2)$ take the following form

$$G_{(a,b)} = e^{i(a\mu_+ + b\mu_-)} = (-1)^{ab/2} g_+^a g_-^b \quad , \quad (a,b) \in \mathbb{Z}_2 \times \mathbb{Z}_2$$

- They admit a two-dimensional rep

$$G_{(0,0)} = \mathbb{I}_2 \quad , \quad G_{(1,0)} = \sigma_1 \quad , \quad G_{(0,1)} = \sigma_3 \quad , \quad G_{(1,1)} = \sigma_2$$

- Vector space = qubit Hilbert space

Projection condition to define a qubit: $S_\pm = g_\pm^2 = 1$

Gottesman-Kitaev-Preskill Code

The embedding of qubits is essentially the well-known GKP code!

Encoding a qubit in an oscillator*

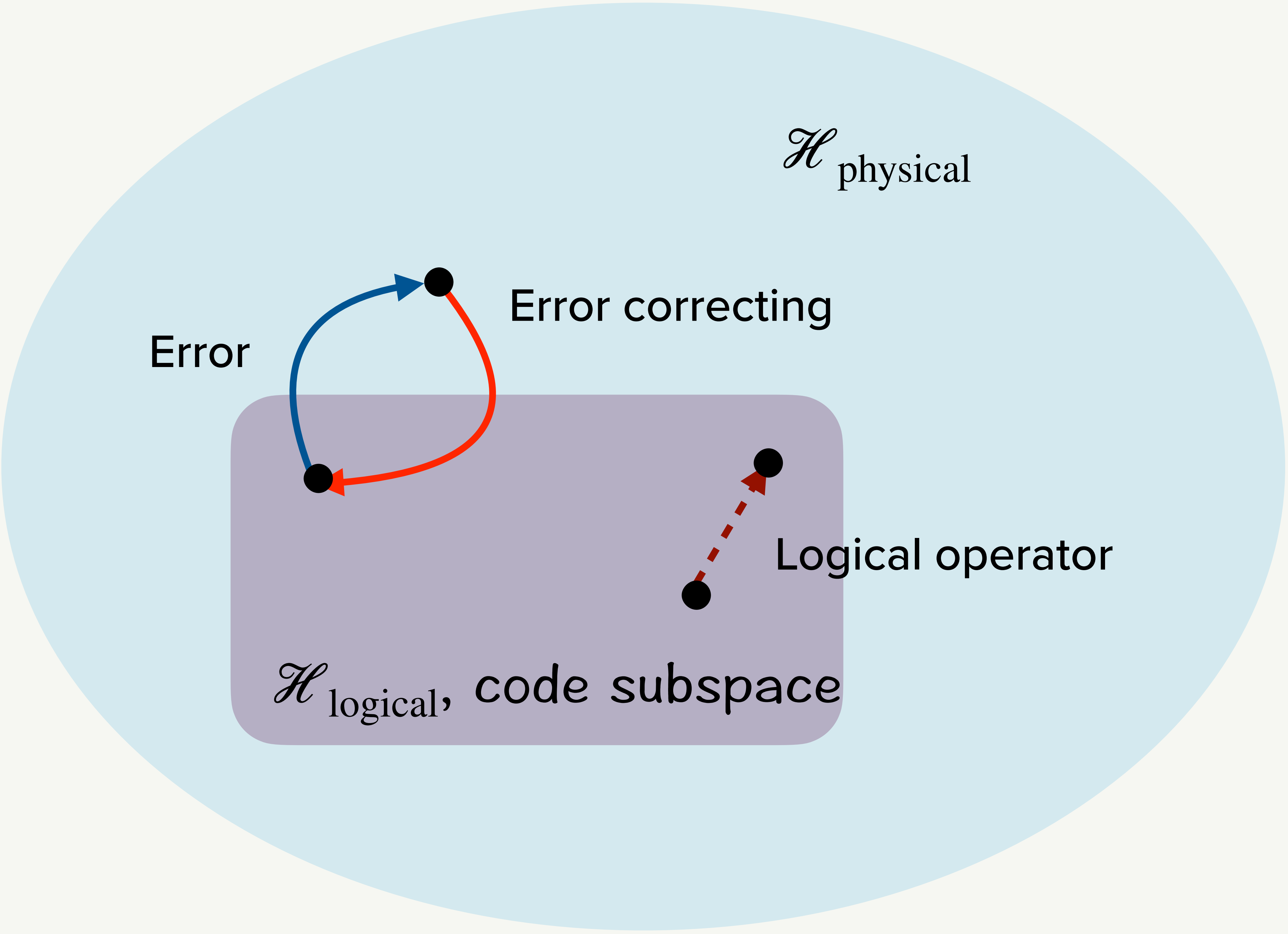
Daniel Gottesman,[(1,2)†] Alexei Kitaev,[(1)‡] and John Preskill[(3)§]
[(1)] *Microsoft Corporation, One Microsoft Way, Redmond, WA 98052, USA*
[(2)] *Computer Science Division, EECS, University of California, Berkeley, CA 94720, USA*
[(3)] *Institute for Quantum Information, California Institute of Technology, Pasadena, CA 91125, USA*

- Embed a finite-dimensional code space in the infinite-dimensional Hilbert space
- Exploit the noncommutative geometry of phase space to protect against errors that shift the values of the canonical variables

[Gottesman, Kitaev, Preskill, '00]

Gottesman-Kitaev-Preskill Code

Our Construction	Stabilizer Code
exponential operators $g_{\pm}$	X,Z gates
$g_{\pm}^2$	stabilizer of the code
qubit condition $g_{\pm}^2 = 1$	code subspace condition
bosonized operators $G_{(a,b)}$	logical (Pauli) operators
vertex operators $e^{i(P_i z_i + \bar{P}_i \bar{z}_i)}$	errors



We will see how this explicitly works shortly..

Symmetries: continuous

- Symmetries for 2-oscillator system: linear transformations that preserve

$$\left[\mu_\alpha^{(k)}, \mu_\beta^{(j)}\right] = i\tau\,\delta^{k+j}\,\epsilon_{\alpha\beta} \qquad k, j = \pm\frac{1}{2} \quad \longrightarrow \quad \mathrm{Sp}(4,\mathbb{R})$$

- Kähler manifold: equipped not only the metric but also the symplectic structure

$$ds^2 = \Omega_{ij}\, dz^i \odot d\bar{z}^j \quad , \quad \Sigma^0 = \Omega_{ij}\, dz^i \wedge d\bar{z}^j$$

- $\mu_\alpha^{(k)}$: two types of indices → two subgroups of $\mathrm{Sp}(4,\mathbb{R})$ naturally follow

$$\mathrm{GL}(2)_{\text{left}} : \quad \mu_\alpha^{(k)} \mapsto (\Lambda_\alpha)^k{}_l\, \mu_\alpha^{(l)} \quad , \quad \alpha = -\,, + \quad ,$$

$$\mathrm{GL}(2)_{\text{right}} : \quad \mu_\alpha^{(k)} \mapsto (\Lambda^{(k)})_\alpha{}^\beta\, \mu_\beta^{(k)} \quad , \quad k = -\frac{1}{2}, \frac{1}{2} \quad .$$

Symmetries: continuous

- This exercise tells us: α and k indices can be viewd as a $\mathrm{SL}(2) \subset \mathrm{GL}(2)$ weight measured by

$$\bar{L}_0 = \frac{1}{i\tau} \sum_{k=\pm 1/2} : \mu^{(k)}_{(+}\mu^{(-k)}_{-)} : \quad ,$$
$$L_0 = \frac{1}{i\tau} \sum_{k=\pm 1/2} k : \mu^{(k)}_{+}\mu^{(-k)}_{-} :$$

The normal ordering:
μ_+ sits at the left of μ_-

- $\mathrm{GL}(2)_{\mathrm{right}}$ also preserve R^2
  - $\mathrm{GL}(2)_{\mathrm{right}} \subset$ Lorentz group
  - $\mathrm{GL}(2)_{\mathrm{right}}$ preserve the Kähler structure

Call $\mathrm{GL}(2)_{\mathrm{right}}$ as Kähler transformations
$\mathrm{GL}(2)_{\mathrm{left}}$ as non-Kähler transformations

Symmetries: discrete

Symmetries for 2-qubit system: must also preserve

$$\boxed{g_\pm = e^{i\mu_\pm} \quad , \quad g_+ g_- \;=\; e^{i\tau} g_- g_+ \quad , \quad \tau = \pi} \qquad \& \qquad \boxed{g_\pm^2 = 1}$$

$$\mathrm{Sp}(4,\mathbb{R}) \;\mapsto\; \mathrm{Sp}(4,\mathbb{Z})$$

also allows transformations such that $\tau \to (2m+1)\pi \;\;,\;\; m \in \mathbb{Z}$

$\Rightarrow$ the full discrete Lorentz transformations are symmetries of 2-qubit system

Clifford Group

symmetry for a 2-qubit system = Clifford group

- Pauli operator acting on the Hilbert space: $G_{(a,b)} \otimes G_{(c,d)}$ (Pauli string)
- Clifford group: a subgroup of $U\left(2^2\right)$ which preserves this factorization

Given the symplectic rep of the Pauli string, the Clifford group exactly describes the 2-qubit symmetry

$$G_{(a,b)} = e^{i(a\mu_+ + b\mu_-)} = (-1)^{ab/2}\, g_+^a\, g_-^b \ , \ (a,b) \in \mathbb{Z}_2 \times \mathbb{Z}_2$$

Clifford group is finitely generated by three unitary gates

[Gottesman, Kitaev, Preskill, '00]

CNOT = Lorentz transformation

- Controlled-NOT (CNOT) gate: only one involves entangling the 2-qubit system

$$|a\rangle \otimes |b\rangle \;\mapsto\; |a\rangle \otimes |a+b\rangle$$

'Control' 'Target'

$$|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \;,\; |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

- It flips the 2nd qubit only when the 1st one is 'activated'

$$x_{\alpha\dot{\alpha}} = \begin{pmatrix} z_1 & -\bar{z}_2 \\ z_2 & \bar{z}_1 \end{pmatrix}$$

$$\mathrm{CNOT} \;:\; \begin{pmatrix} z_1 & -\bar{z}_2 \\ z_2 & \bar{z}_1 \end{pmatrix} \mapsto \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} z_1 & -\bar{z}_2 \\ z_2 & \bar{z}_1 \end{pmatrix}$$

$SL(2,\mathbb{R})_{\text{right}} \to SL(2,\mathbb{Z})_{\text{right}}$ of half of the Lorentz group

[Gottesman, '98]
[Gottesman, Kitaev, Preskill, '00]

Clifford Gates

- Non-kähler transformations: $\mathrm{SL}(2,\mathbb{R}) \to \mathrm{SL}(2,\mathbb{Z})$ — symmetry group of a single qubit
- Hadamard or Fourier gate (F)

$$F \;:\; \begin{pmatrix}\mu_-\\ \mu_+\end{pmatrix} \mapsto \begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}\begin{pmatrix}\mu_-\\ \mu_+\end{pmatrix}$$

- Phase gate (P)

$$P \;:\; \begin{pmatrix}\mu_-\\ \mu_+\end{pmatrix} \mapsto \begin{pmatrix}1 & -1\\ 0 & 1\end{pmatrix}\begin{pmatrix}\mu_-\\ \mu_+\end{pmatrix}$$

[Gottesman, '98]

[Gottesman, Kitaev, Preskill, '00]

Outline

From Fuzzy Spacetimes to Qubits

From Qubits to Qunits

"n": N-dimensional Hilbert space

Celestial CFT from Quantum Error Correction

From Qubits to Qunits

- Consider a field $\tilde{\mu}_\alpha(x^+)$ valued in $\mathbb{R}^2$ defined along the x^+ cycle of Klein space.

$$\tilde{\mu}_\alpha(x^+) = \sum_{k=\pm 1/2} \tilde{\mu}_\alpha^{(k)} e^{ikx^+} \rightarrow \tilde{\mu}_\alpha(x^+) = \sum_{k\in\frac{1}{2}+\mathbb{Z}} \tilde{\mu}_\alpha^{(k)} e^{ikx^+}$$

Modes $\tilde{\mu}_\alpha^{(k)}$ generalize the notion of $z, \bar{z}$ coordinates

- The emergence of N-dimensional Hilbert spaces (qunits) follows from truncating the series (assume even N)

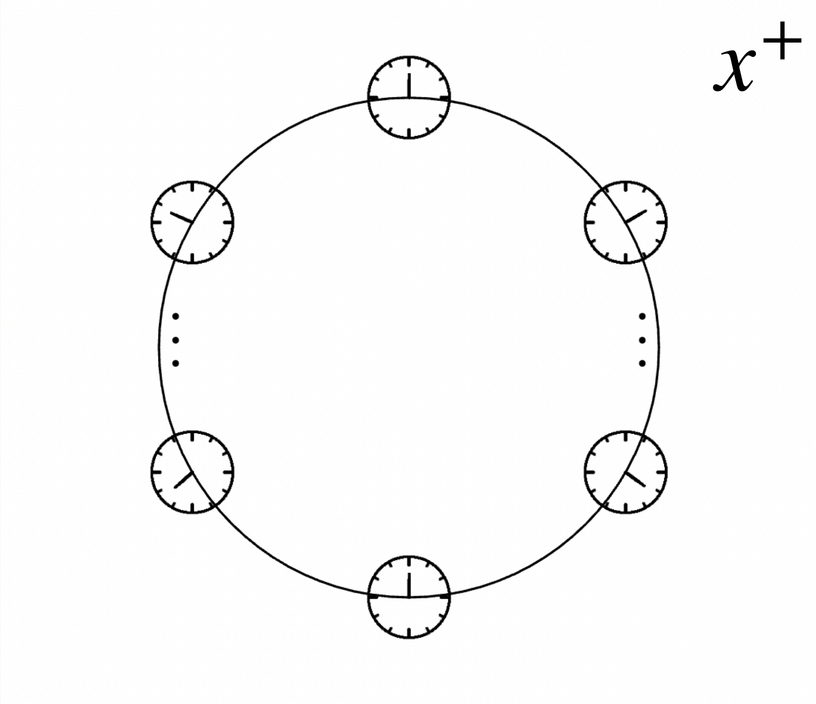
$$\tilde{\mu}_\alpha(x^+) = \sum_{k\in\frac{1}{2}+\mathbb{Z}} \tilde{\mu}_\alpha^{(k)} e^{ikx^+} \rightarrow \sum_{k=-\frac{N-1}{2}}^{\frac{N-1}{2}} \tilde{\mu}_\alpha^{(k)} e^{ikx^+}$$

From Qubits to Qunits

- Truncation of the series can be realized by discretizing the torus

Define the Hilbert space by imposing $\left[\tilde{\mu}_-(x_j^+), \tilde{\mu}_+(x_k^+)\right] \; = \; i\tau\,\delta_{jk}$

$$x_j^+ \;=\; \frac{2\pi j}{N}$$
$$j = 0, \ldots, N-1$$



$$\left[\tilde{\mu}_-^{(k)}, \tilde{\mu}_+^{(j)}\right] \;=\; i\tilde{\tau}\,\delta^{j+k} \quad , \quad \tilde{\tau} \;=\; \frac{\tau}{N} \quad , \quad k, j = -\frac{N-1}{2}, \ldots, \frac{N-1}{2}$$

2-qubit to N-qunit

Recall that $\left[\xi_i, \bar{\xi}_j\right] \;=\; -\,i\tilde{\tau}\,\delta_{ij} + O(1/R^4) \;,\; \tilde{\tau} \;=\; \frac{\tau}{R^2} \;\Rightarrow\; N \propto R^2$

A single qunit GKP

$$\left[\mu_\alpha, \mu_\beta\right] \;=\; i\,\tau\,\epsilon_{\alpha\beta} \;\;,\;\; \tau \;=\; \frac{2\pi}{N} \;\;,\;\; N \in 2\mathbb{Z}_+$$

- Introduce the unitary operator

$g_+ g_- = e^{i\tau} g_- g_+$

$S_\pm$ commute with $g_\pm$

$$g_\pm \;:=\; e^{i\mu_\pm} \quad \text{and} \quad S_\pm := \; g_\pm^N$$

- Code subspace: $S_\pm = 1$

simultaneously diagonalize $S_\pm$ and one of $g_\pm$ (Z)

$$|0\rangle \xrightarrow{X} |1\rangle \xrightarrow{X} |2\rangle \xrightarrow{X} \cdots \xrightarrow{X} |N-1\rangle$$

$$X : |i\rangle \to |i+1\rangle$$

- Logical operators: $g_\pm$ (analog of σ_1 and σ_3 in qubit) ✓ what else?

Most general operator: $$G_\lambda \;=\; \exp i\left(\lambda^+\mu_+ + \lambda^-\mu_-\right) \;=\; e^{i\,[\lambda\mu]}$$

A single qunit GKP

- Displacement operators satisfy the Weyl algebra

$$G_{\lambda_1} G_{\lambda_2} \;=\; e^{i\frac{\tau}{2}[\lambda_1 \lambda_2]} G_{\lambda_1+\lambda_2} \;=\; e^{i\tau[\lambda_1\lambda_2]} G_{\lambda_2} G_{\lambda_1}$$

These operators form a closed algebra of dimension N^2

- Logical operators:

$$[G_\lambda, S_\pm] \;=\; 0 \;\Longrightarrow\; (\lambda_+, \lambda_-) \;\in\; \mathbb{Z}_N \times \mathbb{Z}_N$$ "Stabilizer lattice"

- Write a basis of $\mathscr{H}_N$
  - Two different vacua: $|0\rangle_\pm \;=\; \sum_{p,q\in\mathbb{Z}_N} g_\pm^p S_\mp^q |r\rangle$ which satisfy $g_\pm |0\rangle_\pm = |0\rangle_\pm$
  - Construct the qunit by acting with $g_\mp$: $|n\rangle_\pm \;=\; g_\mp^n |0\rangle_\pm \;\;,\;\; n = 0,\ldots,N-1$

$\mathrm{SL}(2,\mathbb{R})$ symmetry is spontaneously broken

A single qunit GKP

- Go beyond the code subspace

error syndrome

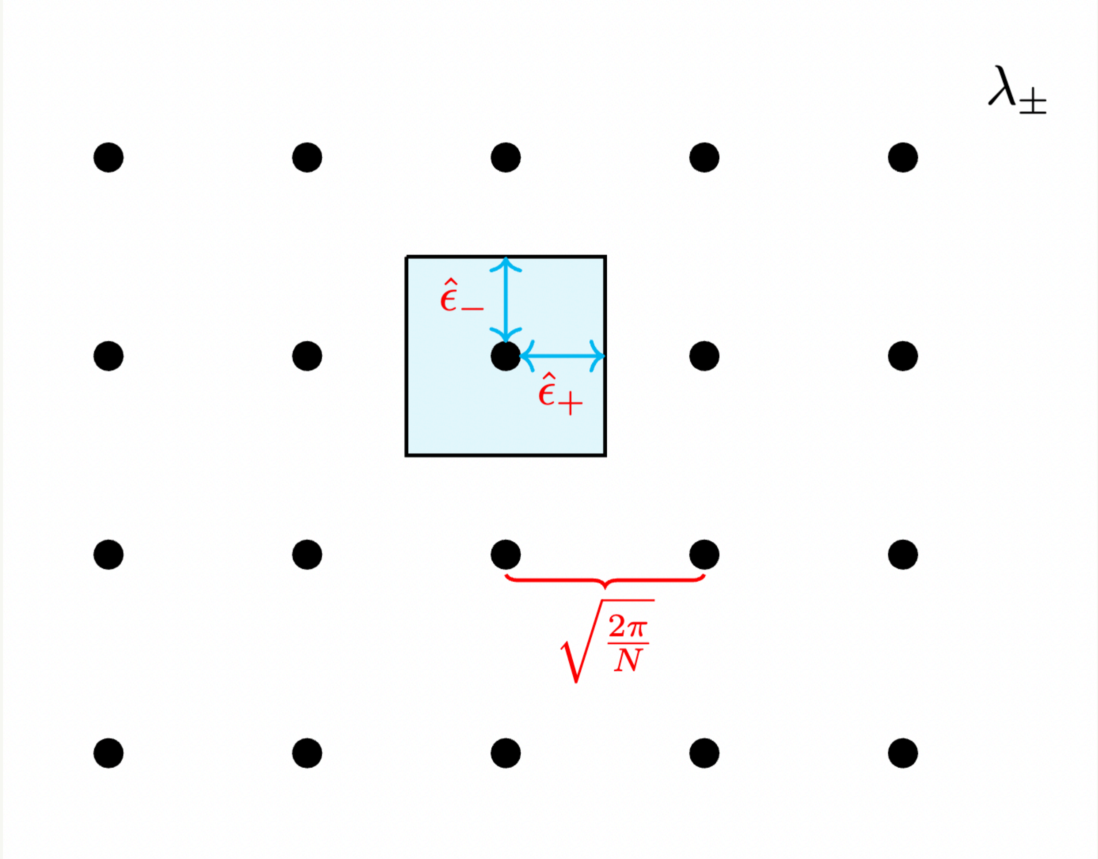
$$S_\pm |\psi\rangle_s \;=\; e^{iNs_\pm}|\psi\rangle_s \qquad s = (s_+, s_-) \qquad s_\pm \;\sim\; s_\pm \;+\; \frac{2\pi}{N}$$

- Errors acting on our code subspace

$$E_{\hat{\epsilon}} \;=\; \exp i\Big(\hat{\epsilon}^+\mu_+ + \hat{\epsilon}^-\mu_-\Big)$$

- They can be measured by evaluating the stabilizer since they shift the $s_\pm$

$$s'_\pm \;=\; s_\pm \;+\; \frac{2\pi}{N}\,\hat{\epsilon}_\pm$$



Errors can be diagnosed when $|\Delta\hat{\epsilon}_\pm| \;<\; \frac{1}{2}\sqrt{\frac{2\pi}{N}}$

Small errors ↔ soft spacetime fluctuations

N-qunit System

$$\left[\mu_\alpha^{(k)}, \mu_\beta^{(l)}\right] = i\,\tau\,\epsilon_{\alpha\beta}\,\delta^{k+l} \quad , \quad -\frac{N-1}{2} \leq k, l \leq \frac{N-1}{2}$$

- Code is constructed in the same way:

$$g_\pm^{(k)} := e^{i\mu_\pm^{(k)}} \quad \text{and} \quad S_\pm^{(k)} := (g_\pm^{(k)})^N$$

- Logical operators

$$G_{\lambda_{-\frac{N-1}{2}}}^{(-\frac{N-1}{2})} \otimes \ldots \otimes G_{\lambda_{\frac{N-1}{2}}}^{(\frac{N-1}{2})} = \exp\left(i \sum_{k=-\frac{N-1}{2}}^{\frac{N-1}{2}} [\lambda_k \mu^{(k)}] \right)$$

- Symmetry: $\mathrm{Sp}(2N, \mathbb{R}) \to \mathrm{Sp}(2N, \mathbb{Z})$

hints on the CFT side..

Continuous Symmetries

Two natural subgroups: $\mathscr{V}_{\text{left}}$, $\mathscr{V}_{\text{right}}$

- only transform k index
- $\mathrm{GL}(N)$ $\quad \mathscr{M} = \mathscr{M}_{kj} : \mu_+^{(k)} \mu_-^{(j)} :$
- Discrete (truncated) Virasoro:

$$L_m \;:\; \mathscr{M}_{kj} = \delta_{k+j-m} \left(\frac{m}{2} - k \right)$$

$$\left[L_m, \mu_\alpha^{(k)} \right] = - \left(\frac{m}{2} + k \right) \mu_\alpha^{(k+m)}$$

Check our paper for more interesting discussions on it!

Truncated: only $2N-3$ of them act nontrivially

- only transform α index
- $N/2$ copies of $\mathrm{GL}(2)_{\text{right}}$ Kähler transformation
- $w_{\alpha\beta}^{(2)} = \sum_k : \mu_{(\alpha}^{(k)} \mu_{\beta)}^{(-k)} : \;\; \in \mathscr{V}_{\text{right}}$

Generalize to nonlinear generators: w_∞

Left (non-Kähler): Virasoro

Right (Kähler): a color $w_{1+\infty} \approx U(N \to \infty)$

Outline

From Fuzzy Spacetimes to Qubits

From Qubits to Qunits

Celestial CFT from Quantum Error Correction

Emerge as $N \to \infty$

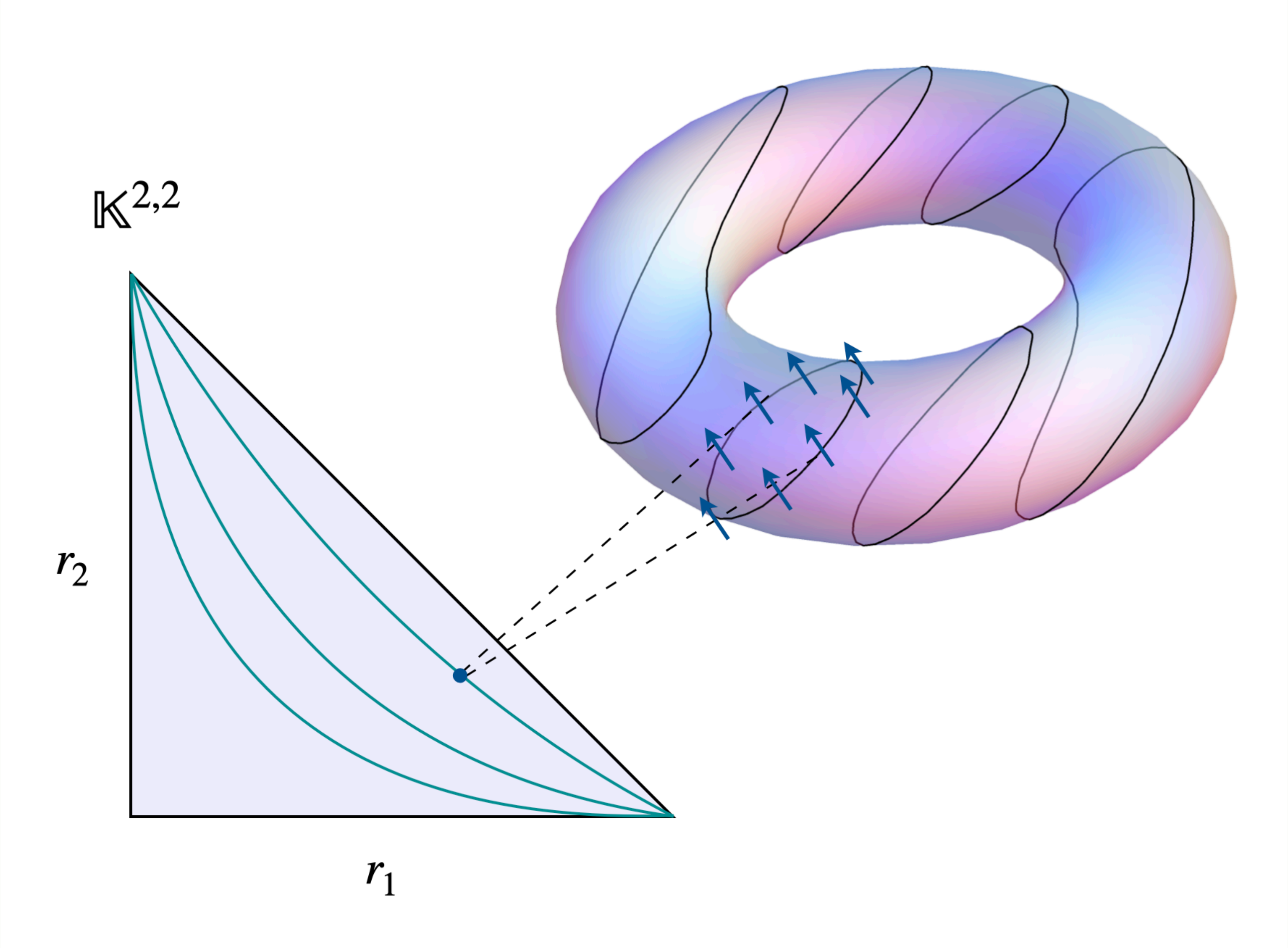
Holographic code: from qunits to CCFT

- To construct a holographic code, we insert N-qunits along the x^+-cycle

$$\left[\mu_\alpha^{(k)}, \mu_\beta^{(l)}\right] = i\,\tau\,\epsilon_{\alpha\beta}\,\delta^{k+l}$$

we allow τ to flow as $\tau \propto 1/N, 1/R^2$

- As approaching to the null bndy $R^2 \to \infty$ and $N \to \infty$, the N-qunit system is anticipated to flow towards a CFT as its continuum limit



Next: 1. Under large N limit, how CFT emerges

2. Physical meanings of the code: physical states, logical states, and errors

CFT from large N

- Consider $N \to \infty$
  - The truncated Virasoro becomes the actual Viraroro $L_m \to \sum_{k \in \mathbb{Z}+1/2} \left(\frac{m}{2} - k\right) : \mu_+^{(k)} \mu_-^{(m-k)} :$
  - The oscillator field $\mu_\alpha(z) = \sum_{k=-\frac{N-1}{2}}^{\frac{N-1}{2}} \frac{\mu_\alpha^{(k)}}{z^{k+\frac{1}{2}}} \to \mu_\alpha(z) = \sum_{k \in \mathbb{Z}+1/2} \frac{\mu_\alpha^{(k)}}{z^{k+\frac{1}{2}}}$
  - Define $T(z) \equiv \sum_{m \in \mathbb{Z}} \frac{L_m}{z^{m+2}} = \frac{1}{2} \epsilon^{\alpha\beta} : \mu_\alpha(z) \, \partial \mu_\beta(z) :$

    which is the stress-energy tensor of a holomorphic CFT: the $c = 1/2$ free fermion!

$\left[\mu_\alpha^{(k)}, \mu_\beta^{(l)}\right] = i\tau \epsilon_{\alpha\beta} \delta^{k+l}$ is equivalent to the OPE $\mu_\alpha(z_1)\, \mu_\beta(z_2) \sim \frac{i\tau}{z_{12}} \epsilon_{\alpha\beta}$

- The internal $U(N)$ becomes $Lw_{1+\infty}$

$$w^{(p)}_{\alpha_1 \ldots \alpha_p}(z) = : \mu_{(\alpha_1}(z) \cdots \mu_{\alpha_p)}(z) :$$

[Adamo, Mason, Sharma, '21A, '21B]

[Guevara, '22]

Supertranslation states

- Code states: under $N \to \infty$

Mode expansion!

with chiral weight $h = 1/2$

$$G_{\lambda_{-\frac{N-1}{2}}}^{(-\frac{N-1}{2})} \otimes \ldots \otimes G_{\lambda_{\frac{N-1}{2}}}^{(\frac{N-1}{2})} \quad \to \quad \mathcal{G}_\eta \;=\; \exp\left(i \sum_k [\eta^{(k)} \mu^{(-k)}] \right) \;=\; \exp\left(i \oint \frac{dz}{2\pi i} [\eta(z) \mu(z)] \right)$$

- Logical states satisfy $\eta_\alpha^{(k)} \in \mathbb{Z}$

Code states = supertranslation eigenstates (hard states)

protect hard states with quantized soft hair

- Supertranslation charge action: $\left[P_{m\alpha}, \mathcal{G}_\eta\right] \;=\; i\,\eta_\alpha^{(m)}\, \mathcal{G}_\eta$

Supertranslation charge carried by $\mathcal{G}_\eta$

Momentum eigenstates

- $\mathscr{G}_\eta = \exp\left(i\oint \frac{dz}{2\pi i}[\eta(z)\mu(z)]\right)$ states
  - carry an infinite set of soft hair
  - generalize the momentum states
- How to recover the usual momentum eigenstates?

$$\eta_\alpha(z) = \frac{\tilde{\lambda}_\alpha}{z-w} \longrightarrow G_{\tilde{\lambda}}(w) = e^{i\,[\tilde{\lambda}\mu(w)]}$$

Errors

- Generic possible error acting on the code subspace takes the following form

$$E_\kappa = \exp\left(i \sum_j \left[\kappa^{(j)} \mu^{(-j)}\right] \right)$$

- Valid error syndrome requires

$$|\kappa^{(j)\pm}| = \left| \oint \frac{dz}{2\pi i} z^{j-1/2} \kappa^{\pm}(z) \right| < \frac{1}{2} \sqrt{\frac{2\pi}{N}}$$

- Consider $\kappa_{\pm}(z) = \dfrac{\tilde{\lambda}_{\pm}}{z-w}$ → error E_κ = inserting a graviton

QEC Condition: $$|w|^{j-1/2} \omega \sqrt{N} < \sqrt{\frac{\pi}{2}} \ , \quad |\bar{w}| < 1 \ .$$

Error = Soft Insertion

$$|w|^{j-1/2}\omega\sqrt{N} < \sqrt{\frac{\pi}{2}} \; , \; |\bar{w}| < 1 \; .$$

$$|w|,|\bar{w}| < 1 \; , \; \omega \leq \Lambda \; , \; \Lambda = \sqrt{\frac{\pi}{2N}}$$

$N\uparrow \; , \Lambda\downarrow$

Tunable parameters:

- ω: the energy of the inserted graviton state
- $(w,\bar{w})$: location of the insertion
- N: an (even) integer, the dimension of the logical subspace

One can trade off between these parameters

Correctable errors = soft graviton insertions near the hard state

Errors shift the wavefunction via

$$\eta_{\pm}(z) \mapsto \eta_{\pm}(z) + \frac{\tilde{\lambda}_{\pm}}{z-w}$$

w as the pole

Energy ω and $\bar{w}$ can be obtained by measuring the stabilizers

Summary

1. The quantization of the noncommutative geometry (Klein spacetimes) allows us to introduce the machinery of QECC

2. The noncommutative structure $\left[\mu_\alpha^{(k)}, \mu_\beta^{(l)}\right] = i\tau\epsilon_{\alpha\beta}\delta^{k+l}$ → qunit as GKP code

3. Given that the central term τ can be renormalized based on R^2 and N, we postulate the existence of a 'holographic code'

$$V : \mathscr{H}_{bulk} \rightarrow \mathscr{H}_{boundary}$$

N-qunit at R_0

Under $N \rightarrow \infty$, the N-qunit system flows to a CFT (a toy model of CCFT)

Outlook

- IR Divergences?
- Spin Models for Celestial CFT?
- Entanglement entropy?
- ...

Thank YOU!