Geometrization of 1D, N-extended Super-Poincaré Algebra and SUSY Holography Conjecture

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Abstract

Deciphering the mathematical structures in the one-dimensional supersymmetric models that secretly encode the information of higher-dimensional counterparts is one of the key tasks in the SUSY holography conjecture. The graphical representations of 1D, N-extended Super-Poincaré algebra provide a powerful tool.

In this work, a conjecture is made that the weight space for 4D, N-extended supersymmetrical representations is embedded within the permutohedra associated with permutation groups $\mathbb{S}_d$. The fact that Klein's Vierergruppe of $\mathbb{S}_4$ plays the role of Hopping operators provides strong evidence supporting this conjecture. It is shown that the appearance of the mathematics of 4D, N = 1 minimal off-shell supersymmetry representations is equivalent to solving a four-color problem on the truncated octahedron. This observation suggests an entirely new way to approach the off-shell SUSY auxiliary field problem based on IT algorithms probing the properties of $\mathbb{S}_d$.

Motivation

The SUSY Holography conjecture was first proposed in [1] as reduce higher-dimensional supersymmetric models to one dimension, and one-dimensional models encode the structure of their higher-dimensional counterparts. In this program, the key object to study is the one-dimensional models.

Graphs have demonstrated an unexpected power to "clear out the mathematical underbrush" encountered by theoretical physicists. Feynman Diagrams are a spectacular example of this. While we make no similar claims about the breadth of possible impacts from the developments which began with the recognition of the existence of the $\mathcal{GR}(d, \text{N})$ ("Garden Algebras") as a foundation of supersymmetric representation theory and their evolution into the introduction of adinkras [2], we do hold that adinkras provide similarly important tools within the domain of the representation theory of supersymmetrical theories. One hint about this involves the pathways that adinkras have opened from supersymmetrical theories, including field theories, to error-correction codes.

Adinkras	Representation of $\mathfrak{po}^{1\|N}$
Vertex bipartition	Bosonic/Fermionic bipartition
Colored edges by I	Action of $Q_{\rm I}$
Dashing	Sign in $Q_{\rm I}$
Change of rank	Power of ∂_τ
Rank function	Engineering dimension

Adinkras

1D, N-extended Super-Poincaré algebra is generated by the time translation generator ∂_τ and N supercharges $Q_{\rm I}$ (I$= 1, \dots,$N), satisfying

$$\{Q_{\rm I}, Q_{\rm J}\} = 2i\delta_{\rm IJ}\partial_\tau \ \ , \ \ [Q_{\rm I}, \partial_\tau] = [\partial_\tau, \partial_\tau] = 0 \ \ . \qquad (1)$$

The generic representation of this algebra, also called supermultiplet in physics, is formed by a set of fields $\{\phi_A, \psi_B\}$, where $A, B = 1, \dots, d$. ϕ_A is a bosonic field while ψ_B denotes a fermionic field. They are related to each other via supersymmetry transformations which can be expressed by the following.

$$Q_{\rm I}\,\phi_A(\tau) = c\,\partial_\tau^\lambda \psi_B(\tau) \ \ , \ \ Q_{\rm I}\,\psi_B(\tau) = \frac{i}{c}\,\partial_\tau^{1-\lambda}\phi_A(\tau) \ \ , \qquad (2)$$

where $c = \pm 1$ and $\lambda = 0, 1$. Then, there are four types of configurations in total. Start from (2), one can define a "*Adinkra*" to graphically represent a supermultiplet following the rules shown in the table below.

Action of Q_I	Adinkra	Action of Q_I	Adinkra
$Q_I \begin{bmatrix} \psi_B \\ \phi_A \end{bmatrix} = \begin{bmatrix} i\dot{\phi}_A \\ \psi_B \end{bmatrix}$	B, I, A	$Q_I \begin{bmatrix} \psi_B \\ \phi_A \end{bmatrix} = \begin{bmatrix} -i\dot{\phi}_A \\ -\psi_B \end{bmatrix}$	B, I, A
$Q_I \begin{bmatrix} \phi_A \\ \psi_B \end{bmatrix} = \begin{bmatrix} i\dot{\psi}_B \\ \phi_A \end{bmatrix}$	A, I, B	$Q_I \begin{bmatrix} \phi_A \\ \psi_B \end{bmatrix} = \begin{bmatrix} -i\dot{\psi}_B \\ -\phi_A \end{bmatrix}$	A, I, B

Valise Adinkra is a special class of adinkras in which all bosons sit in the same level and all fermions sit in the same level. SUSY transformation laws encoded by valise adinkras can be described completely by L adjacent matrices as

$$\text{D}_{\rm I}\,\phi_A = i(\text{L}_{\rm I})_{AB}\,\psi_B \ \ , \ \ \text{D}_{\rm I}\,\psi_B = (\text{R}_{\rm I})_{BA}\,(\partial_\tau\phi_A) \ \ , \ \ \text{R}_{\rm I} = (\text{L}_{\rm I})^{\rm T} \ \ . \qquad (3)$$

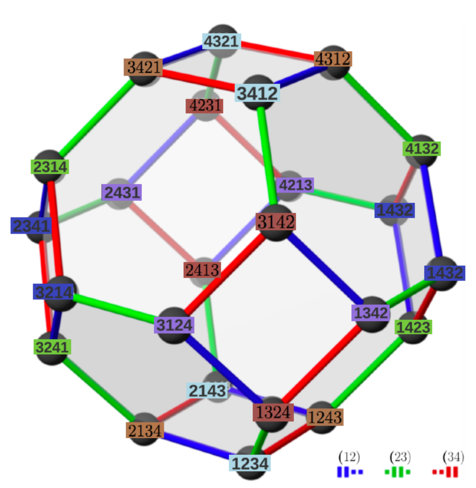
These N L/R adjacent matrices consequently satisfy the so-called "Garden Algebras" $\mathcal{GR}(d, \text{N})$,

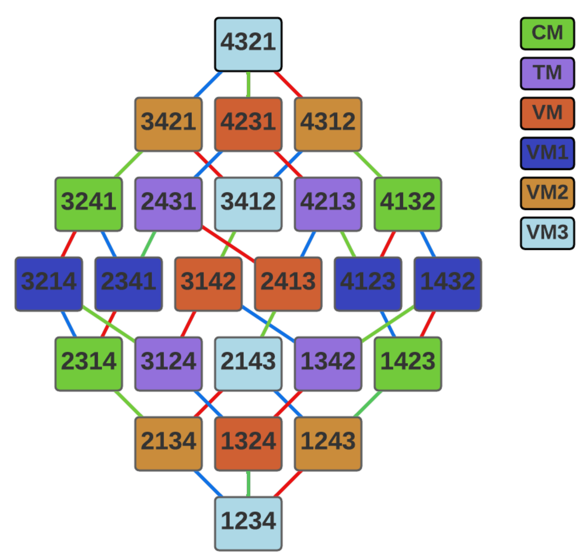
$$\text{L}_{\rm I}\,\text{R}_{\rm J} + \text{L}_{\rm J}\,\text{R}_{\rm I} = 2\delta_{\rm IJ}\,\mathbb{I}_d \ \ , \ \ \text{R}_{\rm I}\,\text{L}_{\rm J} + \text{R}_{\rm J}\,\text{L}_{\rm I} = 2\delta_{\rm IJ}\,\mathbb{I}_d \ \ . \qquad (4)$$

N=4: Permutohedron

When N=4, the minimal supermultiplets contain 4 bosons and 4 fermions, namely $d = 4$ and L-matrices are 4×4. In order to decipher the mathematical structure that contains higher-dimensional information, a good starting point is to study the *chromotopology* of adinkras, where we always set $c = 1$. Then the L adjacent matrices are elements of the symmetric group S_4.

One can use permutohedron to visualize S_4. Consider the projection from 4D, $\mathcal{N} = 1$ SUSY to 1D, N$= 4$ SUSY, a dissection of $\mathbb{S}_4$ is required to be consistent with SUSY [3]. We would like to find out how this can be uncovered without the use of supersymmetry-based arguments, relying solely on properties of $\mathbb{S}_4$. This would lead to a better understanding of which mathematical structures inside adinkras secretly contain information from higher dimensions, which provides a possibility to obtain insights for higher-dimensional theories from 1D statements.



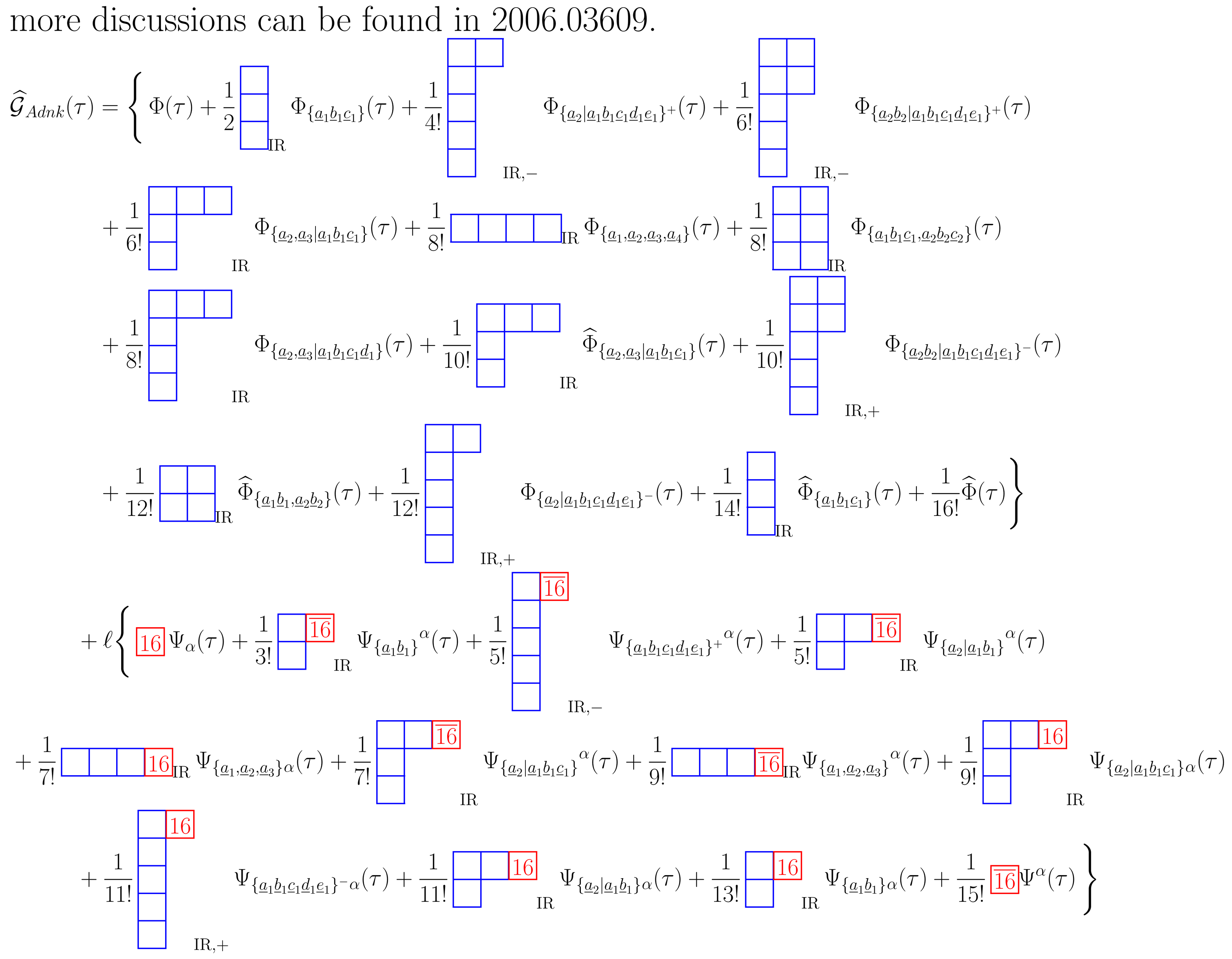


Klein's Vierergruppe as Hopping Operators In this work, one of the main statements is that the elements of Klein's Vierergruppe serve as Hopping operators.

$\mathcal{H}_1 = ()$ $\mathcal{H}_2 = (12)(34)$ $\mathcal{H}_3 = (23)(12)(34)(23)$ $\mathcal{H}_4 = (23)(12)(34)(23)(12)(34)$

Next stop of this research program would be considering N=8 case, corresponding to 4D, $\mathcal{N} = 2$ SUSY, and identifying the Hopping operators in the "Omnitruncated 7-simplex" which has 40,320 nodes and 141,120 edges.

1D, N=16 Adynkrafield from projection: It contains 32,768 bosons and 32,768 fermions. It also contains the information associated with the Lorentz representations (via the Young Tableaux) of the original 10D, $\mathcal{N} = 1$ scalar supermultiplet for which it is the **hologram**! Definitions of irreducible bosonic/spinorial YT's and more discussions can be found in 2006.03609.



References

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