4D Gamma Matrix

- tensor product
- real rep of gamma matrices in 4D
  - Clifford algebra
  - identies
- spinor metric, two form, basis
- Fierz Identities

Pauli matrices

$$(\sigma^1)_\alpha{}^\beta = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad (\sigma^2)_\alpha{}^\beta = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad (\sigma^3)_\alpha{}^\beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

su(2) $\quad S_i = \frac{1}{2}\sigma_i \qquad [S_i, S_j] = i\epsilon_{ijk} S_k$

properties:

① $\sigma^i \sigma^j = i\epsilon^{ijk}\sigma^k$, $\quad i \neq j$

② $(\sigma^i)^2 = \mathbb{I}_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

③ $\{\sigma^i, \sigma^j\} = \sigma^i\sigma^j + \sigma^j\sigma^i$

$$= 2\delta^{ij}\,\mathbb{I}_2$$

" Clifford Algebra "

δ^{ij} — "metric"

convention

i, j: $\quad \delta_{ij} \quad \delta^{ij}$

m, n: $\quad \eta_{mn}, \ \eta^{mn}$

μ, ν: $\quad g_{\mu\nu}, \ g^{\mu\nu}$

taking trace on ③ :

$$\text{RHS} = \mathrm{Tr}\left[2\delta^{ij}\,\mathbb{I}_2\right] = 4\delta^{ij}$$

$$\text{LHS} = \mathrm{Tr}\left[\{\sigma^i, \sigma^j\}\right] = \mathrm{Tr}\left[\sigma^i\sigma^j + \sigma^j\sigma^i\right]$$

$$= \mathrm{Tr}\left[\sigma^i\sigma^j\right] + \mathrm{Tr}\left[\sigma^j\sigma^i\right] = 2\,\mathrm{Tr}\left[\sigma^i\sigma^j\right]$$

$$\Longrightarrow \mathrm{Tr}\left[\sigma^i\sigma^j\right] = 2\delta^{ij}$$

※ $\mathrm{Tr}[AB] = \mathrm{Tr}[BA]$

$$\mathrm{Tr}[ABC] = \mathrm{Tr}[BCA] = \mathrm{Tr}[CAB]$$

cyclic property of trace.

$$\begin{cases} \check{\gamma}^0 = i\sigma^2 \\ \check{\gamma}^1 = \sigma^3 \\ \hat{\gamma}^2 = \sigma^1 \end{cases}$$

$$\{\check{\gamma}^i, \check{\gamma}^j\} = 2\check{\eta}^{ij}\, \mathbb{I}_2$$

$i, j = 0, 1, 2$

$$\check{\eta}^{ij} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \check{X}^i = \begin{pmatrix} ct \\ x \\ y \end{pmatrix}$$

Minkowski spacetime associated with $\check{\eta}^{ij}$

$$\check{X}_i = (-ct, x, y)$$

Dimensions:

spacetime dimension D : size of the metric

$X^m \quad m = 1, \cdots, D$ (Euclidean)

$0, \cdots, D-1$ (Minkowski)

spinor dimension d : size of the gamma matrices

$d(D)$

$(\gamma^m)_\alpha{}^\beta \quad \alpha, \beta = 1, \cdots, d$

Dirac Equation:

$$(\gamma^m)_\alpha{}^\beta \frac{\partial}{\partial x^m} \Psi_\beta = M \Psi_\alpha$$

M: mass; Ψ: spinor (d)

$$\Psi_\alpha = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \vdots \\ \psi_d \end{pmatrix}$$

Q: $D = 3$ $\quad m = 0, 1, 2$

$\alpha = 1, 2$

D	1	2	3	4	5	6	7	8	9	10	11	12
d	1	2	2	4	4	8	8	16	16	32	32	64
	64	128	128	256								

Bott

※ $(\gamma^m)_{\alpha}{}^{\beta}$: matrix
$\quad\;\;\uparrow\quad\uparrow$

$(\gamma^m)^{\beta}{}_{\alpha}$ ✗

<u>Tensor Product</u> $\otimes$

tensor product of two matrices is defined

$$\begin{pmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{pmatrix} \otimes \begin{pmatrix} q_{11} & q_{12} \\ q_{21} & q_{22} \end{pmatrix} \equiv \begin{pmatrix} P_{11}q_{11} & P_{11}q_{12} & P_{12}q_{11} & P_{12}q_{12} \\ P_{11}q_{21} & P_{11}q_{22} & P_{12}q_{21} & P_{12}q_{22} \\ P_{21}q_{11} & P_{21}q_{12} & P_{22}q_{11} & P_{22}q_{12} \\ P_{21}q_{21} & P_{21}q_{22} & P_{22}q_{21} & P_{22}q_{22} \end{pmatrix}$$

$$\equiv \begin{pmatrix} \boxed{P_{11}Q} & \boxed{P_{12}Q} \\ \boxed{P_{21}Q} & \boxed{P_{22}Q} \end{pmatrix}$$

properties:

① $(P_{m\times m} \otimes Q_{n\times n})$: $(m+n)\times(m+n)$?

matrices

$(mn)\times(mn)$ ✓

② $(A+B)\otimes C = A\otimes C + B\otimes C$

$C\otimes(A+B) = C\otimes A + C\otimes B$

③ $K(A\otimes B) = (KA)\otimes B$

number

$= A\otimes(KB)$

HW

prove:

$$\boxed{\begin{array}{l} ④\quad X = P_{m\times m}\otimes Q_{n\times n} \qquad \underbrace{XY}_{} = (\underbrace{PM}_{})\otimes(\underbrace{QN}_{}) \\ \quad\;\; Y = M_{m\times m}\otimes N_{n\times n} \qquad \text{matrix multiplication} \end{array}}$$

4D: $\quad X^m \; : \; m = 0,1,2,3 \qquad D=4$

$(\gamma^m)_\alpha{}^\beta \qquad \alpha,\beta = 1,2,3,4 \qquad d=4$

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad i\sigma^2 = i\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \qquad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\begin{cases} (\gamma^0)_\alpha{}^\beta = (\sigma^3 \otimes i\sigma^2)_\alpha{}^\beta \\ (\gamma^1)_\alpha{}^\beta = (\mathbb{I}_2 \otimes \sigma^1)_\alpha{}^\beta \\ (\gamma^2)_\alpha{}^\beta = -(i\sigma^2 \otimes i\sigma^2)_\alpha{}^\beta \\ (\gamma^3)_\alpha{}^\beta = (\mathbb{I}_2 \otimes \sigma^3)_\alpha{}^\beta \end{cases}$$

explicitly,

column

row

$$(\gamma^0)_\alpha{}^\beta = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}_\alpha{}^\beta = \begin{pmatrix} \boxed{i\sigma^2}_{2\times 2} & \boxed{0}_{2\times 2} \\ \boxed{0}_{2\times 2} & \boxed{-i\sigma^2}_{2\times 2} \end{pmatrix}_\alpha{}^\beta$$

e.g. $(\gamma^0)_1{}^1 = 0$

$(\gamma^0)_2{}^3 = 0$

homework during the break:

$$(\gamma^1)_\alpha{}^\beta = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}_\alpha{}^\beta \qquad (\gamma^3)_\alpha{}^\beta = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}_\alpha{}^\beta$$

$$(\gamma^2)_\alpha{}^\beta = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}_\alpha{}^\beta$$

check:

$$(\gamma^m)_\alpha{}^\beta (\gamma^n)_\beta{}^\gamma + (\gamma^n)_\alpha{}^\beta (\gamma^m)_\beta{}^\gamma = 2\eta^{mn}\delta_\alpha{}^\gamma$$

$$= 2\eta^{mn}(\mathbb{I}_4)_\alpha{}^\gamma$$

" northwest – southeast "

contraction convention

$$\gamma^m \cdot \gamma^n + \gamma^n \cdot \gamma^m = 2\eta^{mn}\,\mathbb{I}_4$$

(matrix multiplication)

* $m=0,\ n=0$: $\underline{(\gamma^0)_\alpha{}^\beta = \sigma^3 \otimes i\sigma^2}$

$$(P \otimes Q)\cdot(M \otimes N) = (P\cdot M)\otimes(Q\cdot N)$$

$$\text{LHS} = 2(\gamma^0)_\alpha{}^\beta(\gamma^0)_\beta{}^\gamma = 2\left(\sigma^3\otimes i\sigma^2\right)_\alpha{}^\beta\left(\sigma^3\otimes i\sigma^2\right)_\beta{}^\gamma$$

$$= -2\Big[\underbrace{(\sigma^3\cdot\sigma^3)}_{\mathbb{I}_2}\otimes\underbrace{(\sigma^2\cdot\sigma^2)}_{\mathbb{I}_2}\Big]_\alpha{}^\gamma$$

$$= -2\left[\mathbb{I}_2\otimes\mathbb{I}_2\right]_\alpha{}^\gamma = -2(\mathbb{I}_4)_\alpha{}^\gamma = -2\delta_\alpha{}^\gamma$$

$$= 2\eta^{00}\delta_\alpha{}^\gamma \qquad (\eta^{00} = -1)$$

$= \text{RHS}$ ✓

$*$ $m=0$, $n=1$

$$\text{LHS} = (\gamma^0)_\alpha{}^\beta(\gamma^1)_\beta{}^\gamma + (\gamma^1)_\alpha{}^\beta(\gamma^0)_\beta{}^\gamma$$

$$= \left[\sigma^3\otimes i\sigma^2\right]_\alpha{}^\beta\left[\mathbb{I}_2\otimes\sigma^1\right]_\beta{}^\gamma + \left[\mathbb{I}_2\otimes\sigma^1\right]_\alpha{}^\beta\left[\sigma^3\otimes i\sigma^2\right]_\beta{}^\gamma$$

$$= \left[\underbrace{(\sigma^3 \cdot \mathbb{I}_2)} \otimes \underbrace{(i\sigma^2 \cdot \sigma^1)} \right]_\alpha{}^\gamma + \left[(\mathbb{I}_2 \cdot \sigma^3) \otimes (\sigma^1 \cdot i\sigma^2) \right]_\alpha{}^\gamma$$

$$= \left[\sigma^3 \otimes \sigma^3 \right]_\alpha{}^\gamma + \left[\sigma^3 \otimes (-\sigma^3) \right]_\alpha{}^\gamma = 0$$

$$※ \; i\sigma^2 \cdot \sigma^1 = i(i\epsilon^{213}\sigma^3) = i^2(-1)\sigma^3 = \sigma^3$$

$$\text{RHS} = 2\eta^{01} (\mathbb{I}_4)_\alpha{}^\gamma = 2\eta^{01} \delta_\alpha{}^\gamma = 0 = \text{LHS} \quad \checkmark$$

* checking for other m, n

$$(\gamma^5)_\alpha{}^\beta \equiv (i\gamma^0\gamma^1\gamma^2\gamma^3)_\alpha{}^\beta$$

$$= \left[i\left(\sigma^3 \otimes i\sigma^2\right)\left(\mathbb{I}_2 \otimes \sigma^1\right)\left(-i\sigma^2 \otimes i\sigma^2\right)\left(\mathbb{I}_2 \otimes \sigma^3\right)\right]_\alpha{}^\beta$$

Aside: $(P\otimes Q)(M\otimes N) = (P\cdot M)\otimes(Q\cdot N)$

$$(P_1\otimes Q_1)(P_2\otimes Q_2)\cdots(P_N\otimes Q_N)$$

$$= (P_1P_2\cdots P_N)\otimes(Q_1Q_2\cdots Q_N)$$

$$= -\left[\left(\sigma^3\cdot\mathbb{I}_2\cdot\sigma^2\cdot\mathbb{I}_2\right)\otimes\left(\sigma^2\cdot\sigma^1\cdot\sigma^2\cdot\sigma^3\right)\right]_\alpha{}^\beta$$

$(\sigma^3\cdot\sigma^2) = -i\sigma^1$

$-i\sigma^3\cdot\underbrace{\sigma^2\cdot\sigma^3}_{i\sigma^1} = i\sigma^2$

$$(\gamma^5)_\alpha{}^\beta = -\left[(-i\sigma^1)\otimes(i\sigma^2)\right]_\alpha{}^\beta = -\left[\sigma^1\otimes\sigma^2\right]_\alpha{}^\beta$$

※ why γ^5 ? if we call it γ^4, we have $\{\gamma^0, \gamma^1, \gamma^2, \gamma^3, \gamma^4\}$

confuses with $D=5$ gamma matrices.

<u>Homework:</u>

① $(\gamma^5)_\alpha{}^\beta (\gamma^5)_\beta{}^\gamma = ?$

② $(\gamma^5)_\alpha{}^\beta (\gamma^m)_\beta{}^\gamma = K (\gamma^m)_\alpha{}^\beta (\gamma^5)_\beta{}^\gamma$

$m = 0, 1, 2, 3$

solve for K.

Define two-form (0,2) tensor

$$A_{pq} = -A_{qp}$$

$$(\sigma^{mn})_\alpha{}^\beta \equiv \frac{i}{2}\left[(\gamma^m)_\alpha{}^\delta(\gamma^n)_\delta{}^\beta - (\gamma^n)_\alpha{}^\delta(\gamma^m)_\delta{}^\beta\right]$$

$$= \frac{i}{2}\left([\gamma^m, \gamma^n]\right)_\alpha{}^\beta$$

$$\sigma^{mn} = -\sigma^{nm} \qquad m, n = 0, 1, 2, 3$$

# of independent σ^{mn} $= 6$

$\sigma^{01}, \sigma^{02}, \sigma^{03}$

$\sigma^{12}, \sigma^{13}, \sigma^{23}$

Summary: $(\gamma^m)_\alpha{}^\beta$, $(\gamma^5)_\alpha{}^\beta$, $(\sigma^{mn})_\alpha{}^\beta$, $(\gamma^5\gamma^m)_\alpha{}^\beta$, $(\mathbb{I}_4)_\alpha{}^\beta$

counting: $\{4\}$ $\{1\}$ $\{6\}$ $\{4\}$ $\{1\}$

total # $= 16 = 4\times 4 = d\times d$

$\Rightarrow$ a basis of 4×4 matrices.

raise/ lower indices

recall: vector indices $m, n = 0,1,2,3$

$$A^m = \eta^{mn} A_n \; ; \; A_m = \eta_{mn} A^n$$

ρ

spinor indices : spinor metric $C_{\alpha\beta}$, $C^{\alpha\beta}$ $\alpha,\beta = 1,2,3,4$

spinor metric $C_{\alpha\beta} = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}_{\alpha\beta} = -i(\sigma^3 \otimes \sigma^2)_{\alpha\beta}$

inverse spinor metric

$C^{\alpha\beta} = \Big(\text{same} \Big)^{\alpha\beta}$

$C^{\alpha\beta} C_{\alpha\gamma} = \delta^{\beta}_{\gamma}$ NOT MATRIX MULTIPLICATION!

$\left[-i(\sigma^3 \otimes \sigma^2)^{\alpha\beta}\right]\left[-i(\sigma^3 \otimes \sigma^2)\right]_{\alpha\gamma}$

$$= (-1)\left[(\sigma^3 \otimes \sigma^2)^{\alpha\beta} (\sigma^3 \otimes \sigma^2)_{\alpha\gamma} \right]$$

$$= (-1)\left[(-1)(\sigma^3 \otimes \sigma^2)^{\beta\alpha} (\sigma^3 \otimes \sigma^2)_{\alpha\gamma} \right]$$

$$= \left[(\sigma^3 \otimes \sigma^2)(\sigma^3 \otimes \sigma^2) \right]^{\beta}{}_{\gamma} = \left[(\sigma^3\sigma^3) \otimes (\sigma^2\sigma^2) \right]^{\beta}{}_{\gamma} = \delta^{\beta}_{\gamma}$$

properties : ① $C_{\alpha\beta} = - C_{\beta\alpha}$

$C^{\alpha\beta} = - C^{\beta\alpha}$

number !

② lower : $\psi_\beta = \psi^\alpha C_{\alpha\beta} = C_{\alpha\beta}\psi^\alpha$

raise : $\psi^\beta = C^{\beta\alpha}\psi_\alpha = \psi_\alpha C^{\beta\alpha}$

you can move $C^{\alpha\beta}$ ($C_{\alpha\beta}$) around.

but you have to stick with this rule!

$(\gamma^m)_\alpha{}^\beta$:

$$(\gamma^m)_{\alpha\beta} = (\gamma^m)_\alpha{}^\gamma C_{\gamma\beta} = C_{\gamma\beta}(\gamma^m)_\alpha{}^\gamma$$

$$(\gamma^m)^{\alpha\beta} = C^{\alpha\gamma}(\gamma^m)_\gamma{}^\beta = (\gamma^m)_\gamma{}^\beta C^{\alpha\gamma}$$

$$(\gamma^m)_{\alpha\beta} = \underbrace{(\gamma^m)^\gamma{}_\beta}_{\text{NOT DEFINED}} C_{\gamma\alpha} \quad \times$$

$$(\gamma^m)_{\alpha\beta} = (\gamma^m)^{\gamma\delta} C_{\gamma\alpha} C_{\delta\beta}$$

$$(\gamma^0)_{\alpha\beta}: \quad (\gamma^0)_{\alpha\beta} = (\gamma^0)_\alpha{}^\gamma C_{\gamma\beta}$$

$$= \left(\sigma^3 \otimes i\sigma^2\right)_\alpha{}^\gamma \left(-i\sigma^3 \otimes \sigma^2\right)_{\gamma\beta}$$

$$= i(-i)\left((\sigma^3\cdot\sigma^3)\otimes(\sigma^2\cdot\sigma^2)\right)_{\alpha\beta}$$

$$= \left(\mathbb{I}_4\right)_{\alpha\beta}$$

$$(\gamma^1)_{\alpha\beta}: \quad (\gamma^1)_{\alpha\beta} = (\gamma^1)_\alpha{}^\gamma C_{\gamma\beta}$$

$$= \left(\mathbb{I}_2 \otimes \sigma^1\right)_\alpha{}^\gamma \left(-i\sigma^3 \otimes \sigma^2\right)_{\gamma\beta}$$

$$= (-i)\left((\mathbb{I}_2\cdot\sigma^3)\otimes(\sigma^1\cdot\sigma^2)\right)_{\alpha\beta}$$

$$= (-i)\left(\sigma^3 \otimes i\sigma^3\right)_{\alpha\beta} = \left(\sigma^3 \otimes \sigma^3\right)_{\alpha\beta}$$

$(\gamma^2)_{\alpha\beta}$:

$$
\begin{aligned}
(\gamma^2)_{\alpha\beta} &= (\gamma^2)_\alpha{}^\gamma \, C_{\gamma\beta} \\
&= -(i\sigma^2 \otimes i\sigma^2)_\alpha{}^\gamma \, (-i\sigma^3 \otimes \sigma^2)_{\gamma\beta} \\
&= -i\big(\underbrace{(\sigma^2 \cdot \sigma^3)}_{i\sigma^1} \otimes \underbrace{(\sigma^2 \cdot \sigma^2)}_{\mathbb{I}_2} \big)_{\alpha\beta} \\
&= (\sigma^1 \otimes \mathbb{I}_2)_{\alpha\beta}
\end{aligned}
$$

$(\gamma^3)_{\alpha\beta}$:

$$
\begin{aligned}
(\gamma^3)_{\alpha\beta} &= (\gamma^3)_\alpha{}^\gamma \, C_{\gamma\beta} \\
&= (\mathbb{I}_2 \otimes \sigma^3)_\alpha{}^\gamma \, (-i\sigma^3 \otimes \sigma^2)_{\gamma\beta} \\
&= -i\big((\mathbb{I}_2 \cdot \sigma^3) \otimes (\sigma^3 \cdot \sigma^2) \big)_{\alpha\beta} \\
&= -i\big(\sigma^3 \otimes (-i\sigma^1) \big)_{\alpha\beta} \\
&= -(\sigma^3 \otimes \sigma^1)_{\alpha\beta}
\end{aligned}
$$

$$(\gamma^0)_{\alpha\beta} = (\mathbb{I}_2 \otimes \mathbb{I}_2)_{\alpha\beta} \qquad (\gamma^2)_{\alpha\beta} = (\sigma^1 \otimes \mathbb{I}_2)_{\alpha\beta}$$

$$(\gamma^1)_{\alpha\beta} = (\sigma^3 \otimes \sigma^3)_{\alpha\beta} \qquad (\gamma^3)_{\alpha\beta} = -(\sigma^3 \otimes \sigma^1)_{\alpha\beta}$$

$$(\gamma^0)_{\alpha\beta} = (\gamma^0)_{\beta\alpha}$$

Homework:

$$(\gamma^1)_{\alpha\beta} = s_1 (\gamma^1)_{\beta\alpha}$$

$$(\gamma^2)_{\alpha\beta} = s_2 (\gamma^2)_{\beta\alpha}$$

$$(\gamma^3)_{\alpha\beta} = s_3 (\gamma^3)_{\beta\alpha}$$

$$(\gamma^5)_{\alpha\beta} = s_4 (\gamma^5)_{\beta\alpha}$$

find s_1, s_2, s_3, s_4

summary:

$$(\gamma^m)_{\alpha\beta} = (\gamma^m)_{\beta\alpha}$$

$$C_{\alpha\beta} = - C_{\beta\alpha}$$

$$(\gamma^5)_{\alpha\beta} = -(\gamma^5)_{\beta\alpha}$$

$$(\gamma^5\gamma^m)_{\alpha\beta} = l_1 (\gamma^5\gamma^m)_{\beta\alpha}$$

$$(\sigma^{mn})_{\alpha\beta} = l_2 (\sigma^{mn})_{\beta\alpha}$$

$$(\gamma^5\gamma^0)_{\alpha\beta} = (\gamma^5\gamma^0)_\alpha{}^\gamma C_{\gamma\beta} = (\gamma^5)_\alpha{}^\delta (\gamma^0)_{\delta\beta}$$

$$= (\gamma^5)_\alpha{}^\delta (\mathbb{I}_4)_{\delta\beta} = (-\sigma^1 \otimes \sigma^2)_{\alpha\beta}$$

$$(\gamma^5\gamma^0)_\alpha{}^\gamma C_{\gamma\beta} = \underline{(\gamma^5)_\alpha{}^\delta} \, \underline{(\gamma^0)_\delta{}^\gamma C_{\gamma\beta}} \Rightarrow \underline{(\gamma^0)_{\delta\beta}}$$

$$(\gamma^5)_\alpha{}^\delta = (-\sigma^1 \otimes \sigma^2)_\alpha{}^\delta$$

$$(\gamma^0)_\delta{}^\gamma = (\sigma^3 \otimes i\sigma^2)_\delta{}^\gamma$$

$$C_{\gamma\beta} = -i(\sigma^3 \otimes \sigma^2)_{\gamma\beta}$$

$$\begin{aligned}
(\gamma^5\gamma^0)_{\alpha\beta} &= (\gamma^5)_\alpha{}^\delta (\gamma^0)_\delta{}^\gamma C_{\gamma\beta} \\
&= (\underset{\smile}{-}\sigma^1 \otimes \sigma^2)_\alpha{}^\delta (\sigma^3 \otimes \underset{\smile}{i}\sigma^2)_\delta{}^\gamma \underbrace{(-i)}(\sigma^3 \otimes \sigma^2)_{\gamma\beta} \\
&= -\left((\sigma^1 \cdot \sigma^3 \cdot \sigma^3) \otimes (\sigma^2 \cdot \sigma^2 \cdot \sigma^2)\right)_{\alpha\beta} \\
&= -(\sigma^1 \otimes \sigma^2)_{\alpha\beta}
\end{aligned}$$

$$\Rightarrow (\gamma^5\gamma^0)_{\alpha\beta} = -(\gamma^5\gamma^0)_{\beta\alpha}$$

$$(\gamma^5\gamma^m)_{\alpha\beta} = (\gamma^5)_\alpha{}^\gamma (\gamma^m)_{\gamma\beta}$$

$$= (\gamma^5)_\alpha{}^\gamma (\gamma^m)_{\beta\gamma}$$

$$= (\gamma^m)_{\beta\gamma} (\gamma^5)_\alpha{}^\gamma$$

$$= -(\gamma^m)_\beta{}^\gamma (\gamma^5)_{\alpha\gamma}$$

$(\gamma^5)_{\alpha\gamma} = -(\gamma^5)_{\gamma\alpha}$

$$= (\gamma^m)_\beta{}^\gamma (\gamma^5)_{\gamma\alpha}$$

$$= (\gamma^m\gamma^5)_{\beta\alpha}$$

$\gamma^m\gamma^5 = -\gamma^5\gamma^m$

$$= -(\gamma^5\gamma^m)_{\beta\alpha}$$

Aside: $A_\alpha B^\alpha$

$$= A_\alpha C^{\alpha\beta} B_\beta$$

$$= A_\alpha(-C^{\beta\alpha}) B_\beta$$

$$= -\underbrace{C^{\beta\alpha} A_\alpha}\, B_\beta$$

$$= -A^\beta B_\beta$$

$$= -A^\alpha B_\alpha$$

※ $(\gamma^m)_{\gamma\beta} (\gamma^5)_\alpha{}^\gamma$

$$= -\cancel{(\gamma^m)^\gamma{}_\beta} (\gamma^5)_{\alpha\gamma}$$

$(\sigma^{mn})_{\alpha\beta} = K\,(\sigma^{mn})_{\beta\alpha}$ find K

$(\ )_{\alpha\beta}$:

$\alpha, \beta = 1, 2, 3, 4$

$M_{\alpha\beta}$ $\{16\}$

sym $(\alpha\beta)$: $\frac{4\times5}{2} = 10$

antisym $(\alpha\beta)$: $\frac{4\times3}{2} = 6$

$(\ )^{\alpha\beta}$

table is valid as well

$(\gamma^m)_{\alpha\beta}$ $\{4\}$ symmetric on α, β

$(\gamma^5)_{\alpha\beta}$ $\{1\}$ antisymmetric on α, β

$(\gamma^5\gamma^m)_{\alpha\beta}$ $\{4\}$ antisymmetric . .

$C_{\alpha\beta}$ $\{1\}$ antisymmetric . .

$(\sigma^{mn})_{\alpha\beta}$ $\{6\}$ symmetric . .

total # $= 16$

# of sym : $4+6 = 10$

# of antisym : $1+4+1 = 6$

① $\gamma^m \gamma_m = 4\mathbb{I}_4$ recall: $\gamma_m = \eta_{mn}\gamma^n$

② $\gamma^m \gamma^n \gamma_m = -2\gamma^n$

③ $\gamma^m \sigma_{pq} \gamma_m = 0$

④ $\gamma^m \gamma^5 \gamma^n \gamma_m = 2\gamma^5\gamma^n$

⑤ $\gamma^m \gamma^5 \gamma_m = -4\gamma^5$

① $\gamma^m \gamma_m = \eta^{mn}\gamma_n\gamma_m$ hint: $\eta^{mn} = \eta^{nm}$

$$= \eta^{mn}\frac{1}{2}\Big(\underbrace{\{\gamma_n,\gamma_m\}}_{2\eta_{nm}\mathbb{I}_4} + \underbrace{[\gamma_n,\gamma_m]}_{\gamma_n\gamma_m - \gamma_m\gamma_n}\Big)$$

$$= \eta^{mn}\eta_{nm}\mathbb{I}_4 + \frac{1}{2}\eta^{mn}\gamma_n\gamma_m - \frac{1}{2}\underset{\eta^{nm}}{\eta^{mn}}\gamma_m\gamma_n$$

$$= (\eta^{00}\eta_{00} + \eta^{11}\eta_{11} + \eta^{22}\eta_{22} + \eta^{33}\eta_{33})\,\mathbb{I}_4 + \frac{1}{2}\gamma^m\gamma_m - \frac{1}{2}\gamma^n\gamma_n$$

$$= (1+1+1+1)\,\mathbb{I}_4 + \frac{1}{2}\Big(\cancel{\gamma^0\gamma_0 + \gamma^1\gamma_1 + \gamma^2\gamma_2 + \gamma^3\gamma_3} \cancel{- \gamma^0\gamma_0 - \gamma^1\gamma_1 - \gamma^2\gamma_2 - \gamma^3\gamma_3}\Big)$$

$$= 4\,\mathbb{I}_4$$

Aside: $AB = \frac{1}{2}\left(\{A, B\} + [A, B]\right)$

$(\gamma^m)_\alpha{}^\beta$ — use $C_{\alpha\beta}$ to raise/lower

m: use η^{mn} to raise/lower

Clifford algebra (defining property of Gamma matrices)

$$\{\gamma^m, \gamma^n\} = 2\eta^{mn}\,\mathbb{I} = \gamma^m\gamma^n + \gamma^n\gamma^m$$

② $\gamma^m \gamma_n \gamma_m$: hint: we can use $\gamma^m \gamma_m = 4\mathbb{I}_4$

$$\gamma^m \gamma_n \gamma_m = \gamma^m \left(2\eta_{nm} \mathbb{I}_4 - \gamma_m \gamma_n \right)$$

Clifford Algebra

$$= 2\eta_{nm} \gamma^m - \gamma^m \gamma_m \gamma_n$$

$$= 2\gamma_n - 4\mathbb{I}_4 \gamma_n = -2\gamma_n$$

③ $\gamma^m \sigma_{pq} \gamma_m = \gamma^m \frac{i}{2} [\gamma_p, \gamma_q] \gamma_m$

$$= \frac{i}{2} \gamma^m \left(\gamma_p \gamma_q - \gamma_q \gamma_p \right) \gamma_m$$

$$\gamma^m \gamma_p \gamma_q \gamma_m = \gamma^m \gamma_p \left(2\eta_{qm} \mathbb{I}_4 - \gamma_m \gamma_q \right)$$

$$= 2\gamma_q \gamma_p - \underbrace{\gamma^m \gamma_p \gamma_m \gamma_q}$$

$$= 2\gamma_q\gamma_p + 2\gamma_p\gamma_q = 2\{\gamma_q, \gamma_p\} = 4\eta_{qp}\mathbb{I}_4$$

$$\Rightarrow \gamma^m \sigma_{pq}\gamma_m = \frac{i}{2}\left(4\eta_{qp}\mathbb{I}_4 - 4\eta_{pq}\mathbb{I}_4\right) = 0$$

④ $$\gamma^m\gamma^5\gamma^n\gamma_m = -\gamma^5\underbrace{\gamma^m\gamma^n\gamma_m} = -\gamma^5(-2\gamma^n)$$

$$= 2\gamma^5\gamma^n$$

recall: $\gamma^m\gamma^5 = -\gamma^5\gamma^m$

⑤ $$\gamma^m\gamma^5\gamma_m = -\gamma^5\gamma^m\gamma_m = -\gamma^5 4\mathbb{I}_4 = -4\gamma^5$$

$$\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \frac{i}{4!}\epsilon_{mnpq}\,\gamma^m\gamma^n\gamma^p\gamma^q$$

$$= \frac{i}{4!}\Big(\epsilon_{0123}\,\gamma^0\gamma^1\gamma^2\gamma^3 + \epsilon_{1023}\,\gamma^1\gamma^0\gamma^2\gamma^3$$

$$+ \epsilon_{3012}\,\gamma^3\gamma^0\gamma^1\gamma^2$$

$$+ \epsilon_{2301}\,\gamma^2\gamma^3\gamma^0\gamma^1 + \cdots\cdots \Big)$$

- $\epsilon_{1023}\,\gamma^1\gamma^0\gamma^2\gamma^3 = (-\epsilon_{0123})(-\gamma^0\gamma^1\gamma^2\gamma^3) = \epsilon_{0123}\,\gamma^0\gamma^1\gamma^2\gamma^3$

☆ $\gamma^m\gamma^n = -\gamma^n\gamma^m \quad (m \neq n)$

- similiar calculations for all 24 terms.

$$\gamma_m \gamma^5 = \frac{i}{3!} \epsilon_{mnpq} \gamma^n \gamma^p \gamma^q$$

$$\gamma_0 \gamma^5 = \gamma_0 \, i \gamma^0 \gamma^1 \gamma^2 \gamma^3 = i \gamma^1 \gamma^2 \gamma^3 = i \epsilon_{0123} \gamma^1 \gamma^2 \gamma^3$$

$$= \frac{i}{3!} \epsilon_{0mnp} \gamma^m \gamma^n \gamma^p$$

$$\gamma_1 \gamma^5 = i \gamma_1 \gamma^0 \gamma^1 \gamma^2 \gamma^3 = -i \gamma^0 \gamma^2 \gamma^3$$

$$= \frac{i}{3!} \epsilon_{1mnp} \gamma^m \gamma^n \gamma^p$$

<u>Homework:</u> $\gamma_m \sigma^{pq} - \sigma^{pq} \gamma_m = 2i \left(\delta_m^p \gamma^q - \delta_m^q \gamma^p \right)$

$$\Rightarrow -\frac{i}{2} \left(\gamma_m \sigma^{pq} - \sigma^{pq} \gamma_m \right) = \delta_m^{[p} \gamma^{q]}$$

$$\Lambda = \begin{pmatrix} \cosh\Phi & -\sinh\Phi \\ -\sinh\Phi & \cosh\Phi \end{pmatrix} \qquad \check{X} = \begin{pmatrix} ct \\ z \end{pmatrix}$$

$$\check{Y} = \Lambda \check{X} = \begin{pmatrix} ct\cosh\Phi - z\sinh\Phi \\ -ct\sinh\Phi + z\cosh\Phi \end{pmatrix}$$

$$\check{Y} \equiv \begin{pmatrix} c\tilde{t} \\ \tilde{z} \end{pmatrix} \Longrightarrow \begin{cases} c\tilde{t} = ct\cosh\Phi - z\sinh\Phi \\ \tilde{z} = -ct\sinh\Phi + z\cosh\Phi \end{cases}$$

suppose $\Phi \ll 1$

$$\sinh\Phi \longrightarrow \Phi \qquad \cosh\Phi \longrightarrow 1 \qquad \begin{cases} c\tilde{t} = ct - z\Phi \\ \tilde{z} = z - ct\Phi \end{cases}$$

$$\text{Define} \begin{cases} \Delta t \equiv \tilde{t} - t \\ \Delta z \equiv \tilde{z} - z \end{cases} \quad \text{then} \begin{cases} c\Delta t = -\Phi z \\ \Delta z = -ct\Phi \end{cases} \Rightarrow \boxed{\begin{cases} \Delta X^0 = -\Phi X^3 \\ \Delta X^3 = -\Phi X^0 \end{cases}}$$

$$\Delta = \Phi L^{03}$$

$$-\frac{i}{2}\left(\gamma_m \sigma^{pq} - \sigma^{pq}\gamma_m\right) = \delta_m^{[p}\gamma^{q]}$$

$$\Delta^{pq}\gamma_m = -\frac{i}{2}[\gamma_m, \sigma^{pq}] = \frac{i}{2}[\sigma^{pq}, \gamma_m] = \delta_m^{[p}\gamma^{q]}$$

$p=0,\ q=3,\ m=0$: $\quad \Delta^{03}\gamma_0 = \gamma^3 \Rightarrow$

$p=0,\ q=3,\ m=3$: $\quad \Delta^{03}\gamma_3 = -\gamma^0 \Rightarrow$

$$\boxed{\begin{aligned} \Delta^{03}\gamma^0 &= -\gamma^3 \\ \Delta^{03}\gamma^3 &= -\gamma^0 \end{aligned}}$$

$$\boxed{\begin{aligned} L^{03}X^0 &= -X^3 \\ L^{03}X^3 &= -X^0 \end{aligned}}$$

blue v.s. red : different representations

If L^{03} is a infinitesimal Lorentz transformation acting on the spacetime coordinates,

then Δ^{03} must be the Lorentz transformation acting on the γ^m

(this is why we claim that "m" on γ^m is the "vector index",

it performs like X^m under the Lorentz transformation)

$$\Lambda^{03} = \begin{pmatrix} \cosh\Phi & 0 & 0 & -\sinh\Phi \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\sinh\Phi & 0 & 0 & \cosh\Phi \end{pmatrix}$$

$$\check{X} = \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix}$$

$$\Lambda^{03}\check{X} = \check{y}$$

Lorentz transformation along z-axis

$$R^{12} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha & 0 \\ 0 & \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$R^{12}\check{x} = \check{y}$$

rotation about the z-axis

$$R^{23} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos\gamma & -\sin\gamma \\ 0 & 0 & \sin\gamma & \cos\gamma \end{pmatrix}$$

$$R^{31} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\beta & 0 & \sin\beta \\ 0 & 0 & 1 & 0 \\ 0 & -\sin\beta & 0 & \cos\beta \end{pmatrix}$$

$$\Lambda^{01} = \begin{pmatrix} \cosh\Phi & -\sinh\Phi & 0 & 0 \\ -\sinh\Phi & \cosh\Phi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \Lambda^{02} = \begin{pmatrix} \cosh\Phi & 0 & -\sinh\Phi & 0 \\ 0 & 1 & 0 & 0 \\ -\sinh\Phi & 0 & \cosh\Phi & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

do the same analysis (taking $\Phi \to 0$), find the generators,

match with $[\sigma^{pq}, \gamma_m]$

σ^{pq}: Lorentz generators.

Recall: $\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \frac{i}{4!}\epsilon_{mnpq}\gamma^m\gamma^n\gamma^p\gamma^q$

$$\gamma_m\gamma^5 = \frac{i}{3!}\epsilon_{mnpq}\gamma^n\gamma^p\gamma^q$$

$$\frac{i}{2}\gamma^m[\gamma^p,\gamma^q] - \frac{i}{2}[\gamma^p,\gamma^q]\gamma^m$$

$$= 2i\,\eta^{m[p}\gamma^{q]}$$

now: $\gamma^m[\gamma^p,\gamma^q] = \,?$

$$\underbrace{\gamma^m\gamma^p\gamma^q} - \gamma^m\gamma^q\gamma^p =$$

$$= \left(2\eta^{mp} - \gamma^p\gamma^m\right)\gamma^q = 2\eta^{mp}\gamma^q - \gamma^p\gamma^m\gamma^q$$

Aside: $\gamma_{\underline{m}}\gamma^5 = \frac{i}{3!}\epsilon_{\underline{m}npq}\gamma^n\gamma^p\gamma^q$

multiply $\epsilon^{mrst}\gamma_m\gamma^5 = \frac{i}{3!}\epsilon^{mrst}\epsilon_{mnpq}\gamma^n\gamma^p\gamma^q$

$$= -\frac{i}{3!}\delta^{[r}_n\delta^s_p\delta^{t]}_q\gamma^n\gamma^p\gamma^q$$

$$= -\frac{i}{3!}\gamma^{[r}\gamma^{s}\gamma^{t]}$$

$$= -\frac{i}{3!}\Big\{\gamma^r\gamma^s\gamma^t - \gamma^r\gamma^t\gamma^s$$

$$+ \gamma^t\gamma^r\gamma^s - \gamma^t\gamma^s\gamma^r$$

$$+ \gamma^s\gamma^t\gamma^r - \gamma^s\gamma^r\gamma^t\Big\}$$

using Clifford algebra

→ bring all other 5 terms to $\gamma^r\gamma^s\gamma^t$ + η's

$$= -\frac{i}{3!}\Big\{\gamma^r\gamma^s\gamma^t - \gamma^r 2\eta^{ts} + \gamma^r\gamma^s\gamma^t$$

$$+ \ldots$$

$$+ \ldots\Big\}$$

$$\epsilon^{mrst}\gamma_m\gamma^5 = -i\left\{\gamma^r\gamma^s\gamma^t - \eta^{ts}\gamma^r + \eta^{rt}\gamma^s - \eta^{rs}\gamma^t\right\}$$

$$\Rightarrow \boxed{\gamma^r\gamma^s\gamma^t = i\epsilon^{mrst}\gamma_m\gamma^5 + \eta^{ts}\gamma^r - \eta^{rt}\gamma^s + \eta^{rs}\gamma^t}$$

$$\gamma^m\gamma^p\gamma^q = i\epsilon^{nmpq}\gamma_n\gamma^5 + \eta^{pq}\gamma^m - \eta^{mq}\gamma^p + \eta^{mp}\gamma^q$$

$$\gamma^m[\gamma^p,\gamma^q] = \gamma^m\gamma^{[p}\gamma^{q]}$$

$$= i\epsilon^{nm[pq]}\gamma_n\gamma^5 + \cancel{\eta^{[pq]}}\gamma^m - 2\eta^{m[q}\gamma^{p]} \qquad 0$$

$$\cancel{+\eta^{m[p}\gamma^{q]}} \qquad \Downarrow$$

$$\eta^{pq} - \eta^{qp} = \eta^{pq} - \eta^{pq}$$

$$= 2i\epsilon^{nmpq}\gamma_n\gamma^5 + 2\eta^{m[p}\gamma^{q]}$$

Homework: ① $\gamma^5[\gamma^p,\gamma^q] = ?$

② $[\gamma^p,\gamma^q]\gamma^m = ?$

Aside: 2D Euclidean space. $\epsilon^{12} = 1$, $\epsilon^{21} = -1$, others $= 0$

ϵ^{ij}, $\epsilon^{ij}\epsilon_{kl} = c\left(\delta^i_k \delta^j_l - \delta^i_l \delta^j_k\right)$

$i, j, k, l = 1, 2$

choose: $i = k \neq j = l$

LHS $= 1$ RHS $= c$ $\Rightarrow c = 1$

2D Minkowski space ϵ^{mn} $\eta^{mn} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$

$\epsilon^{mn}\epsilon_{pq} = c\left(\delta^m_p \delta^n_q - \delta^m_q \delta^n_p\right)$

$m, n, p, q = 0, 1$

choose $m=p=0$, $n=q=1$

$$\text{LHS} = \epsilon^{01}\epsilon_{01} = -1$$

$$\text{RHS} = C \Rightarrow C = -1$$

$$\epsilon^{01} = 1$$

$$\epsilon_{01} = \epsilon^{01}\eta_{00}\eta_{11} = -1$$

$$\Rightarrow \epsilon^{mn}\epsilon_{pq} = -\delta^{[m}_{p}\delta^{n]}_{q}$$

Homework : 4D Minkowski

① $\epsilon^{mnpq}\epsilon_{rstu} = -\delta^{[m}_{r}\delta^{n}_{s}\delta^{p}_{t}\delta^{q]}_{u}$

② $\epsilon^{mnpq}\epsilon_{mrst} = -\delta^{[n}_{r}\delta^{p}_{s}\delta^{q]}_{t}$

recall: $\mathrm{Tr}[T^a T^b] = 2\delta^{ab}$

for matrices, inner product of A and B as $\mathrm{Tr}(AB)$

Trace properties of gamma matrices

① $\mathrm{Tr}(\gamma^m) = 0$ $\quad$: $\mathrm{Tr}(\gamma^m) = (\gamma^m)_\alpha{}^\alpha = \sum_{\alpha=1}^{4} (\gamma^m)_\alpha{}^\alpha$

☆ $\mathrm{Tr}(P \otimes Q) = \mathrm{Tr}(P)\,\mathrm{Tr}(Q)$

② $\mathrm{Tr}(\gamma^m \gamma^n) = \mathrm{Tr}(\gamma^n \gamma^m) = 4\eta^{mn}$

$\mathrm{Tr}(\gamma^m \gamma^n + \gamma^n \gamma^m) = \mathrm{Tr}(2\eta^{mn} \mathbb{I}_4) = 2\eta^{mn} \cdot 4$

$= 2\,\mathrm{Tr}(\gamma^m \gamma^n)$

$$\textcircled{3}\ \mathrm{Tr}(\gamma^m\gamma^n\gamma^p) = \mathrm{Tr}(\gamma^p\gamma^m\gamma^n)$$

$$\mathrm{Tr}(ABC) = \mathrm{Tr}(BCA) = \mathrm{Tr}(CAB)$$

$$= \mathrm{Tr}(2\eta^{pm}\gamma^n - \gamma^m\gamma^p\gamma^n)$$

$$= \mathrm{Tr}(2\eta^{pm}\gamma^n) - \mathrm{Tr}(\gamma^m\gamma^p\gamma^n)$$

$$= -\mathrm{Tr}(\gamma^m\gamma^p\gamma^n) = -\mathrm{Tr}(\gamma^m 2\eta^{pn} - \gamma^m\gamma^n\gamma^p)$$

$$= \mathrm{Tr}(\gamma^m\gamma^n\gamma^p)$$

hint:

- $(\gamma^5)^2 = \mathbb{I}$
- $\gamma^5\gamma^m = -\gamma^m\gamma^5$

$$\mathrm{Tr}(\gamma^m\gamma^n\gamma^p) = \mathrm{Tr}(\gamma^5\gamma^5\gamma^m\gamma^n\gamma^p) = -\mathrm{Tr}(\gamma^5\gamma^m\gamma^n\gamma^p\gamma^5)$$

$$= -\mathrm{Tr}(\gamma^5\gamma^5\gamma^m\gamma^n\gamma^p) = -\mathrm{Tr}(\gamma^m\gamma^n\gamma^p) = 0$$

④ $\mathrm{Tr}(\gamma^{m_1}\gamma^{m_2}\cdots\gamma^{m_p}) = 0$

p is odd

⑤ $\mathrm{Tr}(\gamma^m\gamma^n\gamma^p\gamma^q)$

idea: $\gamma^m\gamma^n\gamma^p\gamma^q \xrightarrow{\text{Clifford algebra}} -\gamma^n\gamma^p\gamma^q\gamma^m + \cdots$

same

$\xrightarrow{\text{cyclic property}} -\gamma^m\gamma^n\gamma^p\gamma^q + \cdots$

$\Rightarrow 2\,\mathrm{Tr}(\gamma^m\gamma^n\gamma^p\gamma^q) = \cdots$

$$\mathrm{Tr}(\gamma^m\gamma^n\gamma^p\gamma^q) = \mathrm{Tr}\left((2\eta^{mn}\mathbb{I}_4 - \gamma^n\gamma^m)\gamma^p\gamma^q\right) = \mathrm{Tr}(2\eta^{mn}\gamma^p\gamma^q)$$

$$- \mathrm{Tr}(\gamma^n\gamma^m\gamma^p\gamma^q) = \mathrm{Tr}(2\eta^{mn}\gamma^p\gamma^q) - \mathrm{Tr}\left(\gamma^n(2\eta^{mp}\mathbb{I}_4 - \gamma^p\gamma^m)\gamma^q\right)$$

$$= \mathrm{Tr}(2\eta^{mn}\gamma^p\gamma^q) - \mathrm{Tr}(2\eta^{mp}\gamma^n\gamma^q) + \mathrm{Tr}(\gamma^n\gamma^p\gamma^m\gamma^q)$$

$$= 2\eta^{mn}\,4\eta^{pq} - 2\eta^{mp}\,4\eta^{nq} + \mathrm{Tr}(\gamma^n\gamma^p\,2\eta^{mq}) - \mathrm{Tr}(\gamma^n\gamma^p\gamma^q\gamma^m)$$

$$\Rightarrow 2\,\mathrm{Tr}(\gamma^m\gamma^n\gamma^p\gamma^q) = 8\eta^{mn}\eta^{pq} - 8\eta^{mp}\eta^{nq} + 8\eta^{mq}\eta^{np}$$

$$\Rightarrow \mathrm{Tr}(\gamma^m\gamma^n\gamma^p\gamma^q) = 4\left(\eta^{mn}\eta^{pq} - \eta^{mp}\eta^{nq} + \eta^{mq}\eta^{np}\right)$$

⑥ $\mathrm{Tr}(\gamma^5) = 0$

method 1: explict tensor product γ^5

method 2: use $\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3$ +

method 3: $\mathrm{Tr}(\gamma^5) = \mathrm{Tr}(\gamma^0\gamma^0\gamma^5) = \cdots = 0$

⑦ $\mathrm{Tr}(\gamma^5\gamma^m) = -\mathrm{Tr}(\gamma^m\gamma^5) = -\mathrm{Tr}(\gamma^5\gamma^m) = 0$

$\gamma^5\gamma^m = -\gamma^m\gamma^5$

⑧ $\mathrm{Tr}(\gamma^5\gamma^m\gamma^n\gamma^p) = 0$

generalization : $\mathrm{Tr}(\gamma^5\gamma^{m_1}\cdots\gamma^{m_p}) = 0$

p is odd.

⑨ $\mathrm{Tr}(\gamma^5\gamma^m\gamma^n) = \mathrm{Tr}(\gamma^\alpha\gamma^\alpha\gamma^5\gamma^m\gamma^n) = (-1)^3\,\mathrm{Tr}(\gamma^\alpha\gamma^5\gamma^m\gamma^n\gamma^\alpha)$

$\alpha \neq m, n$

$\overset{\text{(cyclic)}}{=} \mathrm{Tr}(\gamma^\alpha\gamma^5\gamma^m\gamma^n\gamma^\alpha) = 0$

Summary: $(\gamma^m)_\alpha{}^\beta$, $(\gamma^5)_\alpha{}^\beta$, $\delta_\alpha{}^\beta$, $(\gamma^5\gamma^m)_\alpha{}^\beta$, $(\sigma^{mn})_\alpha{}^\beta$

$$(\gamma^m)_\alpha{}^\beta(\gamma^n)_\beta{}^\alpha =$$

$$(\gamma^m)_\alpha{}^\beta(\gamma^5)_\beta{}^\alpha =$$

$$(\gamma^m)_\alpha{}^\beta\,\delta_\beta{}^\alpha =$$

$$(\gamma^m)_\alpha{}^\beta(\gamma^5\gamma^n)_\beta{}^\alpha =$$

$$(\gamma^m)_\alpha{}^\beta(\sigma^{pq})_\beta{}^\alpha =$$

$$(\gamma^5)_\alpha{}^\beta\,\delta_\beta{}^\alpha =$$

$$(\gamma^5)_\alpha{}^\beta(\gamma^5\gamma^m)_\beta{}^\alpha =$$

$$(\gamma^5)_\alpha{}^\beta(\sigma^{pq})_\beta{}^\alpha =$$

$$\delta_\alpha{}^\beta(\gamma^5\gamma^m)_\beta{}^\alpha =$$

$$\delta_\alpha{}^\beta(\sigma^{mn})_\beta{}^\alpha =$$

$$(\gamma^5\gamma^m)_\alpha{}^\beta(\sigma^{pq})_\beta{}^\alpha =$$